Theory of Multilevel Methods

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Jinchao Xu

May, 1989

Biographical Sketch

The author was born on June 23, 1961, in Hunan province, People's Republic of China. Right after he finished four years undergraduate study in Xiangtan University in January of 1982, he was admitted to be a graduate student at Beijing University where he obtained his Master's degree in Mathematics in July of 1984. In the rest of the year of 1984, he was a teaching assistant in Department of Mathematics of Beijing University. The author came to Cornell in the Spring of 1985. As a graduate student, he has been a teaching assistant, research assistant and fellow of the Mathematical Sciences Institute. In the spring of 1989, he became an assistant professor at the Pennsylvania State University.

Dedicated, with love, to

Xiaoe (Jenny)

Acknowledgements

My principal debt of gratitude is to my thesis advisor Professor James H. Bramble. It has been a great pleasure for me to work with him over the past four years. I have been very much impressed and influenced by his extraordinary insight, persistence and high standards in research. His patience in reading this rather long thesis is very much appreciated.

I wish to thank Professors Alfred Schatz and Lars Wahlbin, who served as members in my special committee and have been a constant help. I would like to say thanks to Ping Lee for his friendship and to Bruce Wade for reading some parts of this thesis. Appreciation is also expressed to Dr. Joseph Pasciak for providing me some of his TEXfiles, and I enjoyed working with him over these years.

I would like to acknowledge the financial support from the Department of Mathematics and NSF, and also graduate fellowship support for the last two years from the Mathematical Sciences Institute at Cornell. Special thanks go to the Liu Memorial Foundation for their award through the Cornell Graduate School.

I am grateful to my parents for their love and care throughout my life, especially to my deceased father who always encouraged and supported me in my pursuit of knowledge in science. My brother, Chaoqun, has been very supportive to me over the past years and deserves my special appreciation.

Finally, I want to thank my wife, Xiaoe, for her love and affection that has always brought me happiness and inspiration. Her understanding and encouragement has been essential for my completion of this dissertation. This thesis is dedicated to her.

Contents

Preface

This thesis is mainly a summary of joint work with my thesis advisor Professor James H. Bramble. Also some parts of the work have been done jointly with Dr. Joseph Pasciak, especially those presented in Chapter 7. Even though an attempt is made to give a coherent presentation on the theory of multilevel methods, as the dissertation of the author, only those results that are closely related to or independently done by myself are included.

Some results from earlier literature or private communications are also included for completeness. Because of the vastness of the multigrid literature, I must apologize for the possible omission of references to works which may be relevant to this thesis.

Chapter 1

Introduction

1.1 Prologue

Multilevel methods are primarily developed for solving algebraic systems arising from the numerical solutions of partial differential equations. These methods are based on certain multilevel structure embodied in the underlying problem. By effectively using this multilevel structure, multilevel algorithms are proven to be one of the most powerful method in solving algebraic equations in modern times.

Research on multilevel methods has been very active in recent years. A lot of work has been devoted to studying the applicability of the multigrid idea to wide range of problems with different backgrounds. Numerical experiments have shown that the algorithms of multilevel type are very efficient in many applications to linear as well as nonlinear problems.

Historically, the idea of multilevel methods was originated from the numerical solutions of second order elliptic boundary–value problem. As a result, this type of problem has been studied most intensively. Nevertheless the existing theory is still incomplete and many fundamental problems call for further investigation. Undoubtedly, a thorough theoretical understanding of the multilevel method on this special subject is of fundamental importance in the whole theory of multilevel methods.

In this work we present a theory for a number of multilevel methods. Our purpose is to give a coherent theoretical treatment of a variety of problems in this direction by modifying or unifying various existing work, providing solutions to several open questions and proposing a number of new algorithms.

1.2 A Capsule Overview

Roughly speaking, there are mainly two types of multilevel methods. The first type of multilevel method is the ordinary multigrid method consisting of fine–space smoothing and coarse–space correction. Another type of the method is obtained by using the multilevel structure to construct a preconditioner; the so–called hierarchical basis multigrid preconditioner is a typical example.

Ordinary multigrid algorithms

The first type multilevel method is what we usually call the multigrid method, to which most of this thesis is devoted. The history of this method is not very long but the literature is abundant. We will not attempt to give a complete overview of the field, instead we will only focus on those papers that have a close relationship to the present work.

The first multigrid work known to the author is the 1961 paper by Fedorenko [42] who proposed a two grid iteration. Soon after, the same author extend his algorithm to multigrid case in [43] and he also presented a convergence proof for Poisson's equation in a square. Some generalization of the method after Fedorenko was first made by Bakhvalov [6] in 1966. He considered more complex finite difference schemes for an equation with variable coefficients and moreover he proposed to use the coarse level solution as a starting value for the multigrid iteration on the next finer level so that the optimal order of complexity of the multigrid algorithm can be achieved. In 1971, Astrakhantsev [2] proved the similar convergence result as Bakhvalov for a finite element discretization.

It seems that multigrid method was not that popular until in 1972 Brandt [29] brought this method to the attention of the western countries. Gradually more work has been done on this work.

Other than the papers mentioned above, early important works on the convergence analysis for multigrid algorithms are, for examples, the papers of Nicolaides [74], Bank and Dupont [10], Hackbusch [46, 47], Bank and Douglas [8]. These theories established the convergence of the algorithms under the condition that the smoothing steps are sufficiently many. However for SPD problems, if the multilevel spaces are nested, numerical experiments show that only one step of smoothing is enough for the convergence. This fact was theoretically justified in the V-cycle proofs of Braess [15, 16], Braess and Hackbusch [18], Maitre and Musy [60], McCormick [69, 70], Verfürth [83] and others. The main idea of these proofs is to analyze the interplay between smoothing and approximate correction directly instead of first developing a two level estimate and then extending it to the W–cycle. In order to get the optimal convergence estimate, it is required to have full regularity (namely $\alpha = 1$ as in (3.5)). But even without the full regularity, slightly weaker estimates are also obtained by Bramble and Pasciak [21] and Decker, Mandel and Parter [39], namely that the reduction number may deteriorate in a logrithmic growth. The arguments of V-cycle proofs are extended to W–cycle, without assuming full elliptic regularity, by Mandel [65], Mandel, McCormick and Ruge [67], Bramble and Pasciak [21]. In [21], Bramble and Pasciak also proposed the so–called variable V-cycle that has convergence properties similar to that of the W–cycle.

For the finite element discretizations of second order elliptic boundary–value problems, in the multigrid literature, attention is largely paid to problems that have a hierarchy of *nested* spaces. As a result, if the underlying domain has a curved boundary, the multigrid algorithm has not been well studied since it is not likely a desired nested sequence of spaces can be constructed on a curved–boundary domain. This will be one of the main issues to be investigated in this thesis.

Another subject which was also not very well understood is the multigrid algorithm for nonsymmetric or indefinite problems. In fact a lot of work has been done on this problem. We refer to Bakhvalov [6], Nicolaides [75], Hackbusch [46], Bank [7] and Mandel [64]. Two types of algorithms are the so called 'symmetric' and 'nonsymmetric' multigrid schemes. The nonsymmetric scheme uses a relaxation procedure based on the original equations whereas the symmetric scheme uses a relaxation based on the symmetric positive definite system associated with the normal equations. For the finite element equations, some results were obtained under rather restrictive assumptions involving the relation between the number of smoothings m and the size of the coarsest grid h_1. For example, Bank [7] gives W-cycle results for both schemes and for arbitrary α which, however, require first that m be sufficiently large and secondly that h_1 be sufficiently small (depending on m). Mandel [64], gives results for the nonsymmetric W-cycle scheme and the V-cycle scheme (assuming full regularity $\alpha = 1$) which are valid for any m if h_1 is chosen sufficiently small (depending on m). But there was one very important question was unanswered, namely if the so–called symmetric scheme is convergent with only one smoothing step. An affirmative answer to this question is given in this thesis.

The theory of the multigrid method is not so well developed if the finite element triangulations are not quasiuniform. This problem has been addressed in many papers but a rigorous mathematical theory is not yet well developed. Important progress in this direction was made by Yserentant in [93] and [92], he analyzed the multigrid algorithm for some special systematically refined meshes that was originally proposed by Babuška, Kellogg and Pitkäranta [5]. But it is not clear how a nested sequence of triangulations can be constructed so that each of them still satisfies the requirement in [5]. Obviously the theory in this direction needs to be further developed. This problem will also be discussed in this work.

Another theoretically interesting problem is how the multigrid algorithms behave for interface problems with considerably large jumps in the coefficients. Numerical experiments demonstrated that with some proper scalings, the algorithm seems to work very well in certain cases. A theoretical justification for this problem was lacking but will be given in this work.

Multigrid analysis for nonconforming finite elements is a rather novel subject. The Crouzeix–Raviart piecewise linear nonconforming element perhaps is the simplest possible element among the family of nonconforming finite element spaces, but its multigrid analysis was still not well–developed. The difficulties lie in the nonnested multilevel spaces and the unnatural prolongation operators. attentions has been largely paid to the construction of appropriate prolongations and some results have been proven under the condition that the number of smoothings is sufficiently large. For the work in this direction, we refer to Brenner [32] and Braess and Verfürth [19]. A new approach is proposed and an optimal result is obtained in this thesis.

The final issue we would like to mention is the parallelization of the multigrid algorithms. As far as we know the multigrid algorithms are recursively defined by levels of spaces and have an optimal efficiency in sequential machines but do not naturally fit into parallel architectures. The trouble is that when the iteration gets to the coarse level which has a very small complexity, most of the processors remain idle. A lot of work has been done in parallelizing the multigrid algorithm, c.f. the references in [49], [50] and [68]. But because of the sequential nature of the algorithm, the obstacle for parallelization seems always there and improvement in implementation is then quite limited. Therefore, if we want a truly parallel multigrid type method, we have to somehow break up the recursive circle of the algorithm. This sounds like a dilemma to the basic multigrid idea, but it is possible with a different approach. In this thesis, we propose a multilevel preconditioning method that is completely parallel.

A more detailed description of the relevant literature on multigrid method will be given as our presentation goes on. At this point we would like to mention a piece of work that is most influential to this work namely the paper by Bramble and Pasciak [21]. In this work there is an innovative approach in formulating the multigrid algorithm. By introducing the so–called *multigrid operators*, a multigrid algorithm can then be characterized by a sequence of recursively defined operators. In this way, notations and presentations are clarified and simplified to a great extent. All the theory on the ordinary multigrid methods in this thesis will be established by means of these multigrid operators.

Multilevel preconditioners

Motivated from the ordinary multigrid method mentioned above, multilevel preconditioners have been constructed by means of multilevel spaces. The first attempt along this line was made by Bank and Dupont [10] where a two level preconditioner was proposed. This idea was extended to multilevel case by Yserentant in [93] and further investigations are made Bank, Dupont and Yserentant in [11]. In this thesis, a new multilevel preconditioner will be presented from a different viewpoint.

1.3 Contents of This Work

Including the current introductory chapter, this thesis contains ten chapters. Chapter 2–6 are essentially the preliminary materials and only the last four chapters, namely Chapter 7–10 make up the body of this thesis which contains the main results on multilevel algorithms.

The theme of the thesis is on the multilevel algorithms for finite element discretizations of second order elliptic boundary–value problems, even though most of our theory will be established in an abstract fashion. It is quite likely that our theory can find applications on other problems (e.g. 4–th or higher order problems, finite difference method, mixed methods etc.), but our focus is only on finite elements. Some such applications may be found in [25].

Roughly speaking, the following problems will be studied in this thesis:

1. Abstract framework of the multigrid algorithm for SPD problems.
2. Unified theory of some known multigrid algorithms.
3. Nonnested multigrid algorithms for curved–boundary–domain problems.
4. Convergence analysis of multigrid algorithms for interface problems and for refined meshes.
5. Multigrid algorithms for nonconforming elements.
6. Convergence analysis of multigrid algorithms for nonsymmetric or indefinite problems.
7. Parallel multilevel preconditioners.

Our presentation is organized by first introducing all of the preliminary materials and then proving all the main theorems of multilevel algorithms by using the preliminary materials. We start from Chapter 2 with basic notation and some results of Sobolev spaces. Then in Chapter 3 and Chapter 4, we introduce various known or new results on finite element spaces that are necessary for the analysis of multilevel algorithms. Chapter 5 gives a brief summary of some iterative methods that are of fundamental importance in multilevel methods. Some technical results for analyzing the convergence of the multigrid algorithm are included in Chapter 6, in particular an approach in studying two level scheme is also included. Chapter 7 then comes to the theory of multigrid methods for symmetric positive definite problems but the presentation there remains abstract. Discussion on some concrete multigrid algorithms begins in Chapter 8. Various problems on the multigrid algorithms for finite element discretizations of SPD elliptic boundary–value problems are studied. Chapter 9 is mainly devoted to a convergence analysis of the multigrid algorithms for nonsymmetric or indefinite elliptic boundary–value problems. Chapter 10 represents the climax of this thesis and a novel parallel multilevel preconditioner is developed.

More detailed descriptions of the contents of this work are given in the following.

Theory of Finite Element Spaces

Generally speaking, a multilevel algorithm is not purely algebraic. Its efficiency strongly depends on the analytic background of the underlying problems. In the applications to finite element methods, a lot of information on the finite element spaces is needed to design and especially to analyze the algorithm. Since finite element application is the main concern of this work, we have included both Chapters 3 and 4 to discuss the properties of finite element spaces.

The material in Chapter 3 is rather standard in the theory of finite element approximations. In particular there is no notion of multilevel spaces in this chapter. We discuss the properties of triangulations (§3.2), finite element spaces (§3.3), inverse inequalities (§3.4), error estimates and stability of nodal value interpolation (§3.6), L^2 projection (§3.8 and §3.9), Galerkin projection (§3.11) etc. The error estimates and stability properties of the orthogonal L^2 projection, which are related to certain simultaneous approximation properties of finite element spaces, are carefully investigated in §3.8. For the purpose of studying interface problems, a certain kind of weighted L^2 projection is proposed and studied in §3.9 and error estimates are obtained in some special situations. Another important notion is the discrete elliptic operators (§3.2), which will play a dominant role in the analysis of multigrid method. The discrete Sobolev norms will be defined in terms of these operators. Theorems on the comparison of discrete Sobolev with the ordinary Sobolev norms are presented (§3.12), which are crucial in the proof of convergence theorems. The discrete elliptic operator is intimately related to the stiffness matrix and their relationship is discussed in §3.3 and their spectral properties are summarized in §3.13.

A theory of finite elements involving multilevel spaces is presented in Chapter 4. The attention is more on the interaction between paire of spaces in the hierarchy of finite element spaces. The core of the theory is the establishment of the so–called regularity and approximation assumption (c.f. (1.2) below or (4.2) in Chapter 4).

The focus of attention of Chapter 4 is in the study of nonnested multilevel spaces, which are obtained from curved–boundary domains. Again we establish the regularity and approximation property. §4.3 concerns specially coupled grids where nonnestedness in the grids only occurs near the boundary, whereas in §4.4 more general nonnested grids are studied.

Some special topics are also discussed in Chapter 4. For example, an estimate for nonquasiuniform meshes is given in §4.2 and an estimate for nonconforming elements is given in §4.5.

Abstract Multigrid Framework for SPD Problems

For the general framework of the multigrid method for symmetric positive definite problems, previous work for variational type problems are summarized in an article by Mandel, McCormick and Bank [66]. In the so-called variational multigrid framework of [66], it is required that

$$A_{k-1}(v,v) = A_k(I_{k-1}^k v, I_{k-1}^k v), \quad \forall v \in \mathcal{M}_k, \tag{1.1}$$

where I_{k-1}^k are *prolongation* operators, A_{k-1} and A_k are the forms on the consecutive spaces $\mathcal{M}_{k-1}$ and $\mathcal{M}_k$. This means that the forms on coarse levels are inherited from the one on the finest level.

Another ingredient in an abstract theory is the so–called *regularity and approximation* assumption, which takes the following form in our notation:

$$|A_k((I - I_{k-1}^k P_{k-1})v, v)| \le C(\rho(A_k)^{-1} {\|A_k v\|_k}^2)^\beta A_k(v,v)^{1-\beta}, \quad \forall v \in \mathcal{M}_k. \tag{1.2}$$

where $\beta \in (0,1]$ is a constant and $\rho(A_k)$ is the spectral radius of A_k.

But, in our experience, we have found that (1.1) and (1.2) seem impossible to be simultaneously satisfied except for simple cases like problems with nested meshes.

In Chapter 7, we will establish a more general theory which allows the constraint (1.1) to be violated. First we replace (1.1) by the following weaker constraint:

$$A_{k-1}(v, v) \geq A_k(I_{k-1}^k v, I_{k-1}^k v), \quad \forall v \in \mathcal{M}_k. \tag{1.3}$$

Under this assumption, together with (1.2), optimal convergence results (with only one smoothing) can be obtained for SPD problems. Secondly, we provide a theory without any constraint like (1.1) or (1.3) at all. In this case ordinary convergence results can only be obtained under the condition that the number of smoothing is sufficiently many. But this is not the point of our theory. We have shown that the variable V–cycle algorithm can provide an optimal preconditioner even though it does not give a reduction in the usual sense. This important feature of multigrid algorithms was not known before. Consequently, in conjunction with the preconditioned conjugate gradient method, multigrid iteration would converge uniformly (with respect to mesh parameters) for problems where only (1.2) is satisfied.

A Trick for Analyzing Two Level Scheme

In the second section of Chapter 6. a technique is included to analyze the convergence of the multigrid algorithm when the elliptic regularity is not easy to verify. Since the assumptions for this approach are easily be satisfied, it is very useful in certain problems where the other approach may fail. Applications can be found in Chapter 8 for interface problems, refined meshes and nonconforming elements.

Multigrid Algorithms for SPD Elliptic Boundary–Value Problems

Multigrid for curved–boundary–domain problems Most of the existing multigrid algorithms which have rigorous theoretical basis are only for those problems that can be furnished with a sequence of nested spaces. Results of this nature will be summarized in §8.2. However if the underlying domain of the problem has a curved boundary, generally speaking it is not likely that a nested sequence of multilevel grids can be constructed. Hence a theory for nonnested spaces is needed.

We will study the multigrid algorithms on two different kinds of triangulations of the domain. The first is very special, that is obtained by successively refining the the grids by a special halving strategy. In this way any two consecutive grids are nested away from the boundary. Since these kinds of grids may be regarded as some sort of perturbation of the nested ones, the corresponding multigrid algorithms have similar convergence properties to the nested multigrid algorithms. As is shown in §8.3.1, by means of the framework in Chapter 7, optimal multigrid convergence results (with only one smoothing) are obtained for the V–cycle, variable V–cycle and W–cycle. Some other types of convergence results are also proven.

Another kind of multilevel triangulation is more general, there are almost no extra unnatural constraints on the triangulations. The framework of Chapter 7 is again applied to study the convergence propertiesi. In particular, we show that the variable V–cycle will provide a preconditioner with the relative condition number being uniformly bounded, see §8.3.1.

In the nonnested multigrid theory the nodal value interpolant is often adopted as a prolongation operator. For general loosely coupled grids, with this choice of prolongation, the convergence theory is currently only rigorously established in two dimensions. We had difficulty extending the theory to three dimensional problems since the desired imbedding properties needed to control the nodal value interpolation for our analysis are lacking. As an alternative approach, we use a new type of operator for prolongation. This operator has stability and approximation properties like the L^2 projection, but its action is relatively easy to evaluate. With this choice of the prolongation, a theory parallel to two dimensional problems is established.

Interface problems and refined meshes Multigrid algorithms with proper scaling for interface problems are analyzed in §8.5. we show that the multigrid algorithm converges uniformly with the jumps on the coefficients provided the number of levels is fixed.

In §8.4, we design an algorithm for quite general nonquasiuniform grids and show an optimal convergence result under the condition that the number of levels is fixed. An important point is to make the right choice of the discrete L^2 product.

Nonconforming elements For the multigrid algorithm for Crouzeix–Raviart nonconforming elements, the approach we will take is different from that in [32] or [19]. Our focus is on the coarse level spaces. In some sense, we think that the coarse level spaces are basically free to be choosen in a multigrid process. In the context of the Crouzeix-Raviart element, there is no reason why one still has to use the nonconforming $\mathcal{P}_1$ element on the coarse levels. Instead we take the conforming $\mathcal{P}_1$ on all the coarse levels. This is the main point in our approach. It turns out that the resulting sequence of spaces are nested and the behavior between any two coarse levels is exactly same as the conforming elements and hence there is nothing more to be analyzed there. The only problem is the transition from the finest space to the next coarser space which is chosen to be the conforming $\mathcal{P}_1$ on the *same* triangulation on which the finest space is defined. We are able to show that the uniform contraction property is still valid between these two grids by using our new technique in Chapter 6. Therefore the whole algorithm is uniformly convergent with only one smoothing step.

Nonsymmetric or Indefinite Problems

In Chapter 9, both the symmetric and nonsymmetric schemes will be analyzed. The result for nonsymmetric schemes (§9.3) is not new; it was given by Mandel [64], but our proof is different and also seems much simpler. The result we will prove in §9.4 are new for the symmetric scheme. We find that the nature of the analysis for these two schemes is different and the symmetric scheme is much harder to handle. The proof we present is very lengthy and technical, but we arrive at almost optimal conclusions. We give results for the V-cycle, variable V-cycle and W-cycle algorithms for any amount of smoothing under the assumption of $\alpha > 3/4$. Our theorems require that h_1 be sufficiently small (independent of the amount of smoothing) and guarantee an iterative convergence rate which is uniformly independent of the number of levels and the mesh size on the finest grid. The assumption that h_1 is sufficiently small is not very restrictive since such an assumption must be made for solvability on the coarsest grid. The results for the V-cycle algorithm are somewhat weaker. We show that the V-cycle converges if h_1 is small enough (depending on the number of levels and α), at a rate which deteriorates as more and more levels are used. Even with the full elliptic regularity assumption, namely $\alpha = 1$, the V-cycle convergence estimates deteriorate like $1 - c/\ln(h^{-1})$.

Parallel Multigrid Preconditioners

Abstract framework Abstract framework of our new preconditioners is described in §10.1 and §10.2.

In §10.1, in an abstract setting, we show that a discrete elliptic operator can be very well preconditioned by a summation of scaled L^2 projections (onto a hierarchy of nested spaces). The corresponding condition numbers are estimated in terms of some a priori assumptions of the underlying multilevel spaces, L^2 projections or Galerkin projections. These assumptions clearly hold in the finite elements context hence the application of this framework is straightforward and is presented in §10.2.

It turns out there is a lot of internal structure in our abstract multilevel framework. This is further investigated in §10.2. Similar to the hierarchical basis preconditioner, our preconditioner can also be obtained from certain decomposition of the approximation space. Our decomposition is based on a sequence of L^2 projections and because of the nestedness of the multilevel spaces, the decomposition is orthogonal in L^2 inner product. Under certain assumptions, we can show that the elliptic operator is spectrally equivalent to a constant operator on each component of the decomposition. In other words the elliptic operator can be preconditioned by that constant operator on each of these components. In this way a preconditioner which is a summation of scaled L^2 projections then results.

The orthogonality property in the resulting preconditioner makes it very flexible in its applications. For example, a power of the preconditioner can be obtained by taking the power of constants on each component, which gives a preconditioner of the corresponding power of the original elliptic operator. This feature leads to an application on preconditioning the $H^{\frac{1}{2}}$ norm on the interfaces in a domain decomposition technique. Such kinds of properties of the new preconditioner are discussed in §10.2.

Application to finite element discretizations More efficient algorithm by using the abstract framework in the concrete case can be obtained by using more structure available in the underlying problem. This is illustrated in §10.3 for the application to finite element equations. In this case, a multilevel finite element spaces $\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_j$ can be obtained by the successively refined triangulations. If we let $\{\phi_i^l\}$ denote the usual nodal basis for the subspace $\mathcal{M}_k$, the preconditioner $\mathcal{B}$ can then be defined by

$$\mathcal{B}v = \sum_{k=1}^{j} h_k^{2-d} \sum_l (v, \phi_k^l)\phi_k^l,$$

where h_k is the mesh size of the grid defining $\mathcal{M}_k$ and d is the dimension of the problem.

Notice that the above preconditioner is simply a double sum, the terms of which can be computed concurrently. The overall complexity of the preconditioner in this parallel implementation is of order $O(jN)$ (where N is the number of unknowns of the approximation space $\mathcal{M}_j$ and j is the number of grid levels). However the algorithm can also be implemented in a recursive fashion so that the number of operations needed in one iteration is of optimal order $O(N)$.

Without assuming any elliptic regularity the relative condition number of this new preconditioner grows at most j^2. But if full elliptic regularity is present, the condition number can be shown to grow only like j.

Our algorithm for finite element equations is similar to the so-called hierarchical basis preconditioner (cf. [11]), but by constrast, our method works equally well in any number of dimensions whereas the hierarchical basis preconditioners deteriorate in three dimensional problems. In addition, the implementation of the hierarchical basis preconditioner is sequential like the ordinary multigrid.

Interface problems In the two dimensional case, the hierarchical basis preconditioner works well even for interface problems with large jumps if the parallelization is not taken into account. For our method we have shown that theoretically it would work as well as hierarchical basis preconditioner as far as convergence propery is concerned, namely the condition number of the resulting preconditioned system only has a logarithmic growth but does not depend on the jumps of the coefficients. Results on three dimensional problems are not very satisfactory yet, but we are still able to establish the convergence theory in a special case that the interfaces do not have interior cross point.

Preliminary Results on Sobolev Spaces

Some other results which are proved in the preliminary chapters may also of theoretical interests themselves. In §2.2, we have an inequality in Soblev space in regard with the relations between $W^{\frac{d}{p},p}(\Omega)$ and $C(\bar{\Omega})$ when the imbedding $W^{\frac{d}{p},p} \hookrightarrow C(\bar{\Omega})$ fails. As its consequences, some discrete Sobolev inequalities can be obtained in finite element spaces (§3.7). Chapter 2 also contains an abstract theorem in Banach space that covers many well-known results or their generalizations on Sobolev spaces, including a Sobolev norm equivalence theorem, and the Bramble–Hilbert Lemma.

1.4 Some Special Notations

Throughout this thesis, we are going to adapt the following notations:

$$x \lesssim y, \quad f \gtrsim g \quad \text{and} \quad u \asymp v$$

means

$$x \leq Cy \quad , \quad f \geq cg \quad \text{and} \quad cv \leq u \leq Cv$$

where C and c, are positive constants independent of the mesh parameters and functions.

Chapter 2

Some Results on Sobolev Spaces

Sobolev spaces are of fundamental importance in the theory of partial differential equations as well as finite element methods. In this chapter, some basic facts on Sobolev spaces are reviewed and some new results of the author are also included which will be used in later chapters.

The outline of this chapter is as follows. Notations and a number of standard results in Sobolev spaces are given in Section 2.1. Section 2.2 is devoted to the proof of a new Sobolev inequality in regard with $W^{\frac{d}{p},p}(\Omega)(p > 1)$ and $C(\bar{\Omega})$. In Section 2.3, we present an abstract theorem in Banach space and derive some useful results in Sobolev spaces as consequences. Section 2.4 contains a trace inequality.

2.1 Notation and Definitions

We will use standard notation for Sobolev spaces, cf. Adams [1]. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary, $L^p(\Omega)$ is a Banach space consisting of $p-$th power integrable functions. The Sobolev space of index (m, p) is defined by

$$W^{m,p}(\Omega) \stackrel{\text{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \leq m\},$$

with a norm

$$\|v\|_{W^{m,p}(\Omega)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha| \le m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$ is multi–integer and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}}, \quad |\alpha| = \sum_{i=1}^d \alpha_i.$$

For $p = 2$, by convention, we denote

$$H^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega).$$

We will have occasion to use the following seminorms:

$$|v|_{W^{m,p}(\Omega)} \stackrel{\text{def}}{=} \left(\sum_{|\alpha| = m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

The fractional Sobolev spaces $W^{m+\sigma,p}(\Omega)(m \ge 0, 0 < \sigma < 1, p \ge 1)$ are defined by the completion of $C^\infty(\Omega)$ in the following norm:

$$\|v\|_{W^{m+\sigma,p}(\Omega)} \stackrel{\text{def}}{=} \left(\|v\|^p_{W^{m,p}(\Omega)} + |v|^p_{W^{m+\sigma,p}(\Omega)} \right)^{\frac{1}{p}}, \tag{2.1}$$

where

$$|v|^p_{W^{m+\sigma,p}(\Omega)} \stackrel{\text{def}}{=} \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x - y|^{d+\sigma p}} dx\, dy.$$

For studying the Dirichlet problem, the following space is essential:

$$H_0^1(\Omega) \stackrel{\text{def}}{=} \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\},$$

where $v|_{\partial\Omega} = 0$ is in the sense of *trace* (c.f [58]). The space $H_0^1(\Omega)$ can also be considered to be the closure of the space of $C_0^\infty(\Omega)$ consisting of compactly supported smooth functions in the topology of $H^1(\Omega)$. More generally, for a measurable subset $\Gamma \subset \partial\Omega$, we denote

$$H^1_\Gamma(\Omega) \overset{\text{def}}{=} \{v \in H^1(\Omega) : v|_\Gamma = 0\}.$$

Similarly, if $\frac{1}{2} < \sigma \leq 1$, we define

$$H^\sigma_\Gamma(\Omega) \overset{\text{def}}{=} \{v \in H^\sigma(\Omega) : v|_\Gamma = 0\},$$

and if $\Gamma = \partial\Omega$

$$H^\sigma_0(\Omega) \overset{\text{def}}{=} \{v \in H^\sigma(\Omega) : v|_{\partial\Omega} = 0\}.$$

Some other Banach spaces will also be used. For example, $C(\bar{\Omega})$ is the space of continuous functions with the usual maximum norm $\|\cdot\|_{C(\bar{\Omega})}$ and for $\lambda \in (0,1]$, $C^{0,\lambda}(\bar{\Omega})$ is the space of the functions satisfying the following Hölder condition

$$\|u\|_{C^{0,\lambda}(\bar{\Omega})} \overset{\text{def}}{=} \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\lambda} < \infty.$$

We quote the following imbedding results (cf. Adams [1]):

$$H^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega) \quad \text{if } d > 2, \tag{2.2}$$

$$W^{s,p}(\Omega) \hookrightarrow C(\bar{\Omega}) \quad \text{if } sp > d. \tag{2.3}$$

Here the symble $\hookrightarrow$ denotes the continous imbedding.

More profoundly,

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\lambda}(\bar{\Omega}), \quad \text{if } sp > d \text{ and } \lambda = \min\{1, s - \tfrac{d}{p}\}. \tag{2.4}$$

For $s = d, p = 1$, (2.3) would still hold, but not for $sp = d$ and $p > 1$, in which case we have

$$W^{\frac{d}{p},p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{if } r < \infty. \tag{2.5}$$

More precisely, by tracing the constants in the proof of the above imbedding, we have

$$\|v\|_{L^r(\Omega)} \lesssim r^{1-\frac{1}{d}} \|v\|_{W^{\frac{d}{p},p}(\Omega)}, \quad \forall v \in W^{\frac{d}{p},p}(\Omega),\ r > 1. \tag{2.6}$$

2.2 An Inequality Relating $W^{\frac{d}{p},p}(\Omega)$ with $C(\bar{\Omega})$

As we pointed out in the foregoing section, the space $W^{d/p,p}(\Omega)$ $(p > 1)$ generally fails to be imbedded into $C(\bar{\Omega})$. The following inequality, due to the author, indicates some relationship between these two spaces. This inequality will also be used in the next chapter to derive some discrete Sobolev inequalities in finite element spaces.

Theorem 2.1 *Assume $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary and $p > 1$, then there exists a constant $C \equiv C(\Omega, d, p)$ such that*

$$\|u\|_{C(\bar{\Omega})} \leq C\{|\log \epsilon|^{1-\frac{1}{d}}\|u\|_{W^{d/p,p}(\Omega)} + \epsilon^{\lambda}\|u\|_{C^{0,\lambda}(\bar{\Omega})}\},$$

for any $\epsilon > 0, \lambda \in (0, 1]$ and $u \in W^{d/p,p}(\Omega) \cap C^{0,\lambda}(\bar{\Omega})$.

A special case of this theorem (actually of Corollary 1 below) for $p = d = 2$ was proven by the author in [85] and announced in [86].

Proof. Without loss of generality, we may assume that $0 < \epsilon \leq 3^{-p}$. Given $u \in W^{d/p,p}(\Omega) \cap C^{0,\lambda}(\bar{\Omega})$, let $x_0 \in \bar{\Omega}$ be such that

$$|u(x_0)| = \|u\|_{C(\bar{\Omega})}.$$

Let $D_\epsilon \equiv \{x \in \Omega : |x - x_0| < \epsilon\}$, then $|D_\epsilon| \equiv meas(D_\epsilon) \gtrsim \epsilon^d$ since $\partial\Omega$ is Lipschitz continuous by hypothesis.

Note, for any $x \in D_\epsilon$, that

$$|u(x_0)| \lesssim |u(x)| + \epsilon^{\lambda}\|u\|_{C^{0,\lambda}(\bar{\Omega})}.$$

Taking the $L^d(D_\epsilon)$-norm on both hand sides of above inequality, we obtain

$$|D_\epsilon|^{\frac{1}{d}}|u(x_0)| \lesssim \|u\|_{L^d(D_\epsilon)} + \epsilon^{\lambda}|D_\epsilon|^{\frac{1}{d}}\|u\|_{C^{0,\lambda}(\bar{\Omega})},$$

hence

$$|u(x_0)| \lesssim \epsilon^{-1}\|u\|_{L^d(D_\epsilon)} + \epsilon^{\lambda}\|u\|_{C^{0,\lambda}(\bar{\Omega})}.$$

An application of Hölder's inequality with $r \geq d$ yields

$$\begin{aligned} \|u\|_{L^d(D_\epsilon)} &\leq |D_\epsilon|^{\frac{1}{d}-\frac{1}{r}}\|u\|_{L^r(\Omega)} \\ &\lesssim \epsilon^{1-\frac{d}{r}}\|u\|_{L^r(\Omega)}. \end{aligned}$$

By (2.5), we have

$$\|u\|_{0,r,\Omega} \leq C(\Omega, d, p) r^{1-\frac{1}{d}}\|u\|_{W^{d/p,p}(\Omega)}, \tag{2.7}$$

for any $r \geq p$ and $u \in W^{d/p,p}(\Omega)$.

Taking $r = \log\frac{1}{\epsilon}$, we get

$$\begin{aligned} \|u\|_{L^d(D_\epsilon)} &\lesssim \epsilon^{1-\frac{d}{r}} r^{1-\frac{1}{d}}\|u\|_{W^{d/p,p}(\Omega)} \\ &\lesssim \epsilon|\log\epsilon|^{1-\frac{1}{d}}\|u\|_{W^{d/p,p}(\Omega)}. \end{aligned}$$

The desired result then follows, completing the proof. ■

Corollary 2.1 *Under the same assumptions as Theorem 2.1, if* $qs > d$ *and* $\lambda = \min\{1, s - \frac{d}{q}\}$, *then*

$$\|u\|_{C(\bar{\Omega})} \lesssim |\log\epsilon|^{1-\frac{1}{d}}\|u\|_{W^{d/p,p}(\Omega)} + \epsilon^{\lambda}\|u\|_{W^{s,q}(\Omega)},$$

for any $\epsilon > 0$ *and* $u \in W^{d/p,p}(\Omega) \cap W^{s,q}(\Omega)$.

Proof. The imbedding (2.4) means that

$$\|u\|_{C^{0,\lambda}(\bar{\Omega})} \lesssim \|u\|_{W^{s,q}(\Omega)}.$$

Applying above Theorem then completes the proof. ■

2.3 An Abstract Theorem with Applications to Sobolev Spaces

There are some results on Sobolev spaces that are particularly useful in the analysis of finite element methods. The aim of this section is to include several results on Sobolev spaces in somewhat generalized forms so as to meet the needs of later applications. We shall first present an abstract theorem on Banach spaces and then apply it to Sobolev spaces.

2.3.1 Abstract Theorem

The theorem to be presented below is an improvement made of the author of a result due to Tartar (unpublished but available in Brezzi and Marini [34], see also Ciarlet [36], Exercise 1. on page 126).

We first need to introduce some new terminology:

Definition 2.1 *Assume that V is a Banach space with a norm $\|\cdot\|_V$ and $F : V \mapsto \mathbb{R}^1_+$ is a functional. We call F a W-norm on V if it satisfies the following two conditions:*

i) *$F(\cdot)$ is a semi-norm on V.*

ii) *If $v_n, v \in V$ and $\lim_{n\to\infty} \|v_n - v\|_V = 0$, then $F(v) \le \lim_{n\to\infty} \sup F(v_n)$.*

And we call F a compact functional on V if for any bounded set $G \subset V$ there exists a sequence $\{v_n\} \subset G$ so that $F(v_n - v_m) \to 0 \quad$ as $n, m \to \infty$.

In the definition of W-norm, ii) is a kind of continuity condition. Seemingly it is even weaker than the usual lower–semi continuity (which requires that $F(v) \le \lim_{n\to\infty} \inf F(v_n)$.). However a nontrivial fact is that any W-norm must be continuous in the strong sense, namely $\lim_{n\to\infty} |F(v_n) - F(v)| = 0$. As a matter of fact, we have

Lemma 2.1 *For any W-norm F on a Banach space V, we have*

$$F(v) \lesssim \|v\|_V, \quad \forall v \in V.$$

The proof of this result is similar to that of *uniform boundedness principle* in Banach space theory (c.f. [78]). The idea is to consider the decomposition $V = \cup_{k=1}^{\infty}\{v \in V : F(v) \leq k\|v\|_V\}$ and apply the Baire category theorem. The details of the proof are omitted here.

Theorem 2.2 *Let V be a Banach space with a norm $\|\cdot\|_V$. Assume there exists a W-norm F and a compact functional T on V such that*

$$\|v\|_V \lesssim F(v) + T(v), \quad \forall v \in V. \tag{2.8}$$

Then the following are true:

i) *For any other W-norm G over V, as long as $\ker(F) \cap \ker(G) = \{0\}$, then*

$$\|v\|_V \eqsim F(v) + G(v), \quad \forall v \in V.$$

ii) *Let $V/\ker(F)$ be the ordinary quotient space and $\|\cdot\|_{V/\ker(F)}$ the corresponding quotient norm, then*

$$\|v\|_{V/\ker(F)} \eqsim F(v), \quad \forall v \in V.$$

iii) *For any other W-norm B over V, as long as $\ker(F) \subset \ker(B)$, then*

$$B(v) \lesssim F(v), \quad \forall v \in V.$$

Proof. First of all, by Lemma (2.1), we immediately have

$$F(v) + G(v) \lesssim \|v\|_V, \quad \forall v \in V.$$

This proves one direction of i). To see the other direction of i), we use a contradiction argument, namely we assume if what we want to show were not true, there would exist $\{v_n\} \subset V$ such that

$$\|v_n\|_V = 1, \quad \text{and} \quad F(v_n) + G(v_n) \to 0 \quad (\text{as } n \to \infty). \tag{2.9}$$

Since $\{v_n\}$ is bounded, by the definition of compact functional, we may assume that

$$T(v_n - v_m) \to 0, \quad \text{as } n, m \to \infty. \tag{2.10}$$

It follows from the hypothesis (2.8), (2.9) and (2.10) that

$$\begin{aligned} \|v_n - v_m\|_V &\lesssim F(v_n - v_m) + T(v_n - v_m) \\ &\leq F(v_n) + F(v_m) + T(v_n - v_m) \to 0, \quad \text{as } n, m \to \infty. \end{aligned}$$

This means that $\{v_n\}$ is a Cauchy sequence on V. But V is a Banach space, hence there exists $v \in V$ so that

$$\lim_{n\to\infty} \|v_n - v\|_V = 0.$$

Since both F and G are W-norms, we conclude that

$$F(v) + G(v) \leq \lim_{n\to\infty} \sup F(v_n) + \lim_{n\to\infty} \sup G(v_n) = 0.$$

Hence $F(v) = G(v) = 0$, i.e., $v \in \ker(F) \cap \ker(G)$. By hypothesis, $v = 0$, but this contradicts (2.9) which implies $\|v\|_V = 1$, completing the proof of i). To prove ii), we should first point out that $\ker(F)$ is obviously a subspace of V. Furthermore we claim that $\ker(F)$ is finite dimensional, in fact, by the hypothesis (2.8), we can easily see that the unit ball in $\ker(F)$ is compact.

We need to show that

$$\inf_{\phi\in\ker(F)} \|v + \phi\|_V \lesssim F(v), \quad \forall v \in V. \tag{2.11}$$

As mentioned above $m \stackrel{\text{def}}{=} \dim(\ker(F)) < \infty$, hence we can choose m functionals $\{f_k : k = 1, 2, \cdots, m\}$ over $\ker(F)$ that forms a basis for the dual space $(\ker(F))^*$. By Hahn-Banach theorem, we may assume that f_k's are all defined on the whole of V. Set

$$G(v) = \sum_{k=1}^{m} |f_k(v)|.$$

Using the fact that $\{f_k\}$ forms a basis of $(\ker(F))^*$, for any $v \in V$, by solving a nonsingular linear system, we can find a $\phi_v \in \ker(F)$ such that

$$f_k(\phi_v) = -f_k(v), \quad k = 1, 2, \cdots, m.$$

Namely

$$G(v + \phi_v) = 0.$$

Since G is obviously a W-norm on V and also $\ker(F) \cap \ker(G) = \{0\}$, we conclude from i) that

$$\begin{aligned} \|v + \phi_v\|_V &\lesssim F(v + \phi_v) + G(v + \phi_v) \\ &= F(v + \phi_v) = F(v). \end{aligned}$$

which implies (2.11) and completes the proof of ii).

Next we are in a position to prove iii). By Lemma 2.1, for any $v \in V$ and $\phi \in \ker(F)$

$$B(v + \phi) \lesssim \|v + \phi\|_V.$$

By hypothesis that $\ker(F) \subset \ker(B)$, we have $\phi \in \ker(B)$, thus $B(v) = B(v + \phi)$, therefore, using ii), we get that

$$B(v) \lesssim \inf_{\phi \in \ker(F)} \|v + \phi\|_V \lesssim F(v),$$

completing the proof. ■

2.3.2 Applications

Even though the abstract theorem presented above is of theoretical interest for its own sake, our main purpose is to apply it to Sobolev spaces.

Given $m \geq 0, 0 < \sigma \leq 1$ and $p \geq 1$, we then take $V = W^{m+\sigma,p}(\Omega)$ and $F(v) = |v|_{W^{m+\sigma,p}(\Omega)}$, $T(v) = \|v\|_{W^{m,p}(\Omega)}$, we can see that (2.8) is trivially satisfied. Obviously, F is a W-norm by definition. By the well–known fact that $W^{m+\sigma,p}(\Omega)$ is compactly imbedded into $W^{m,p}(\Omega)$, T is a compact functional. Also it is straightforward to check that $\ker F = \mathcal{P}_m(\Omega)$. Therefore we can apply Theorem 2.2 to deduce the following (generalized) well-known results:

Theorem 2.3 1. Sobolev Norm Equivalence Theorem

$$\|v\|_{W^{m,p}(\Omega)} \asymp G(v) + |v|_{W^{m+\sigma,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega), \tag{2.12}$$

if G is a W-norm on V such that for $\phi \in \mathcal{P}_m(\Omega)$, $G(\phi) = 0$ iff $\phi = 0$.

2.

$$\inf_{\phi \in \mathcal{P}_m(\Omega)} \|v + \phi\|_{W^{m+\sigma,p}(\Omega)} \asymp |v|_{W^{m+\sigma,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega), \tag{2.13}$$

3. Bramble-Hilbert Lemma

$$B(v) \lesssim |v|_{W^{m+\sigma,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega), \tag{2.14}$$

if B is a W-norm on V such that for all $\phi \in \mathcal{P}_m(\Omega)$, $B(\phi) = 0$.

Results in the above theorem are often presented for $\sigma = 1$ in the literature. In what we called *Sobolev norm equivalence theorem*, G usually takes the form of $G(v) = \sum_{i=1}^{m} |f_i(v)|$, for some bounded linear functionals f_i's. One might find our abstract version with the W-norm more convenient to use.

As we know, the trace of the function in $H^\sigma(\Omega)$ is well-defined if $\sigma > \frac{1}{2}$, hence we can take $G(v) = \int_\Gamma v dx$ in (2.12) and have a special case of (2.12) as follows:

Theorem 2.4 (Poincaré Inequality) *Assume* $\Gamma \subset \partial\Omega$ *such that meas*$(\Gamma) > 0$ *and* $\frac{1}{2} < \sigma \leq 1$, *then*

$$\|v\|_{H^\sigma(\Omega)} \lesssim \left|\int_\Gamma v dx\right| + |v|_{H^\sigma(\Omega)}, \quad \forall v \in H^\sigma(\Omega).$$

Consequently

$$\|v\|_{H^\sigma(\Omega)} \lesssim |v|_{H^\sigma(\Omega)}, \quad \forall v \in H^\sigma_\Gamma(\Omega).$$

2.4 A Trace Inequality

The following simple inequality will be useful later, which was shown to the author by Bramble.

Lemma 2.2

$$\|u\|_{L^2(\partial\Omega)} \lesssim \epsilon^{-1}\|u\|_{L^2(\Omega)} + \epsilon\|u\|_{H^1(\Omega)},$$

for any $u \in H^1(\Omega)$ *and* $\epsilon \in (0, 1)$.

Proof. Taking a function $\vec{\rho} \in [C^1(\bar{\Omega})]^2$ such that

$$\vec{\rho}(x) \cdot \vec{n}(x) \geq 1, \quad \forall x \in \partial\Omega,$$

where $\vec{n}(x)$ is the outer normal direction of $\partial\Omega$ at x, we have

$$\begin{aligned} \int_{\partial\Omega} u^2 dx &\leq \int_{\partial\Omega} u^2 \vec{\rho} \cdot \vec{n} dx \\ &= -\int_\Omega div\vec{\rho} u^2 dx - \int_\Omega 2u\vec{\rho} \cdot \nabla u dx \\ &\lesssim \int_\Omega u^2 dx + \int_\Omega |u| \, |\nabla u| dx \\ &\lesssim (1 + \epsilon^{-2}) \int_\Omega u^2 dx + \epsilon^2 \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

This completes the proof. ■

Chapter 3

Classical Theory of Finite Element Spaces

As we pointed out earlier, the main purpose in this work is to design and analyze multigrid algorithms for finite element discretizations of second order elliptic boundary value problems. In this chapter a summary of various results for the finite element spaces will be given. The materials are only chosen to meet the need for later development. Justifications are included for a number of the results that are well-known but somehow their proofs are hard to locate in the literature. Several new results by the author are also presented.

Without loss of generality, we will confine ourself on the piecewise linear triangular finite elements. The material in this chapter does not have much direct connection with the notion of "multigrid", namely the analysis is more or less only on a single finite element space. The results here will be used to establish a theory for multilevel spaces in next chapter.

The remainder of this chapter is organized as follows. In Section 3.1, a model partial differential equation is described. After introducing the triangulations in Section 3.2, finite element spaces and the some relevant operators are discussed in Section 3.4. Inverse inequalities are summarized in Section 3.4 and in particular an inverse inequality for a fractional Sobolev norm is presented. Properties of the mass matrix in terms of the quasiuniformity are discussed in Section 3.5. In

Section 3.6, some estimates for the nodal value interpolation are given. A discrete Sobolev imbedding theorem is proved in Section 3.7. Section 3.8 is devoted to the study of the stability and approximation property of the orthogonal L^2 projection and as a by-product a simultaneous approximation property is established. A kind of weighted L^2 projection is introduced and studied in Section 3.9. Section 3.10 is concerned with a special operator that is closely related to the L^2 projection. An error estimate and stability estimate for Galerkin projections are presented in Section 3.11. Section 3.12 discusses the equivalence between the discrete norms and the ordinary Sobolev norms. Finally in Section 3.13, some estimates for the condition numbers of the discrete elliptic operators are given.

3.1 Model Partial Differential Equation

We shall consider the problem of approximating the solution U of

$$\begin{aligned} \mathcal{L}U &= F \quad in\, \Omega, \\ U &= 0 \quad on\, \partial\Omega. \end{aligned} \tag{3.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^d$ and $\mathcal{L}$ is given by

$$\mathcal{L}v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j}) + a_0 v, \tag{3.2}$$

with $\{a_{ij}\}$ uniformly positive definite and bounded on Ω and a_0 is nonnegative. The bilinear form corresponding to the operator $\mathcal{L}$ is defined by

$$A(v,w) = \sum_{i,j=1}^{d} \int_\Omega (a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + a_0 vw)dx. \tag{3.3}$$

This form is defined for all v and w in the Sobolev space $H^1(\Omega)$. Clearly, $U \in H_0^1(\Omega)$ is the solution of

$$A(U,\chi) = (F,\chi) \quad \forall \chi \in H_0^1(\Omega). \tag{3.4}$$

We make the following elliptic regularity assumption. There exists a constant $\alpha \in (0,1]$ so that

$$\|U\|_{H^{1+\alpha}(\Omega)} \leq C\|F\|_{H^{\alpha-1}(\Omega)}, \tag{3.5}$$

for the solution U of (3.4), where C is a constant depending on the domain Ω and the coefficients defining $\mathcal{L}$.

3.2 Triangulations

For the given domain $\Omega \subset \mathbb{R}^d$, by a triangulation of Ω we mean a set $\mathcal{T}_k$ of $d-$simplices such that the intersection of any two simplices in $\mathcal{T}$ either consists of a common lower dimensional simplex or is empty and $\Omega_k \stackrel{\text{def}}{=} \cup\{\tau : \tau \in \mathcal{T}_k\}$ is either equal to Ω (if Ω is a tetrahedral) or close to Ω in the sense that

$$\max_{x\in\tau\cap\partial\Omega_k} \text{dist}(x,\partial\Omega) \lesssim (\text{diam } \tau)^2, \quad \forall\tau \in \mathcal{T}_k, \tau\cap\partial\Omega_k \neq \emptyset.$$

Now supposing a family of triangulations $\{\mathcal{T}_k : k \in \mathcal{I}\}$ are given on Ω, we define the following two parameters:

$$h_k = \max_{\tau\in\mathcal{T}_k} h_\tau; \quad \underline{h}_k = \min_{\tau\in\mathcal{T}_k} h_\tau,$$

where h_τ =diam τ.

Concerning the triangulations, we will always make the following basic assumptions:

(A3.1) If ρ_τ denotes the radius of the ball inscribed in τ, then

$$\sup_{k\in\mathcal{I}} \max_{\tau\in\mathcal{T}_k} \frac{h_\tau}{\rho_\tau} \leq \sigma_0.$$

Definition 3.1 *$\{\mathcal{T}_k : k \in \mathcal{I}\}$ is said to be quasi-uniform if it satisfies (***A3.1***) and the following*

(A3.2) $$h_k \lesssim \underline{h}_k.$$

The assumption (**A3.1**) is a local assumption, as is meant by above definition, for $d = 2$ for example, it assures that each triangle will not degenerate into a segment in the limiting case. More explicitly, this assumption is equivalent to the following well-known minimum angle condition:

$$\inf_{k\in\mathcal{I}} \min_{\tau\in\mathcal{T}_k} \theta_\tau \geq \theta_0 > 0 \tag{3.6}$$

where θ_τ is the minimum interior angle of τ for $\tau \in \mathcal{T}_k$ and θ_0 is a constant.

On the other hand, the assumption (**A3.2**) is a global assumption, which says that the smallest mesh size is not too small compared with the largest mesh size of the same triangulation. By the definition, in a quasiuniform triangulation, all the elements are about the same size asymptotically.

The following lemma asserts that the triangles in $\mathcal{T}_k$ do not differ very much locally. This fact will be often used implicitly in our later analysis.

Lemma 3.1 *If the triangulation $\{\mathcal{T}_k, k \in \mathcal{I}\}$ satisfies (**A3.1**), then*

$$\sup_{k\in\mathcal{I}} \max_{\tau,\tau'\in\mathcal{T}_k,\, \tau\cap\tau'\neq\emptyset} \frac{h_\tau}{h_{\tau'}} \lesssim 1.$$

Proof. To illustrate the main idea, we present our proof for $d = 2$.

If τ and τ' have one edge in common, then

$$\frac{h_\tau}{h'_\tau} = \frac{h_\tau}{\rho_\tau}\,\frac{\rho_\tau}{h'_\tau} < \frac{h_\tau}{\rho_\tau}.$$

In general, by (3.6), $\exists l \leq \frac{2\pi}{\theta_0}$ and $\tau_1, \cdots, \tau_l \in \mathcal{T}_k$ such that τ_i and τ_{i+1} has one edge in common, for $i = 0, 1, \cdots, l$ with $\tau_0 = \tau$ and $\tau_l = \tau'$, hence

$$\frac{h_\tau}{h'_\tau} = \prod_{i=0}^{l-1} \frac{h_{\tau_i}}{\rho_{\tau_i}}\,\frac{\rho_{\tau_i}}{h_{\tau_{i+1}}} < \prod_{i=0}^{l-1} \frac{h_{\tau_i}}{\rho_{\tau_i}} \leq \sigma_0^{\frac{2\pi}{\theta_0}}.$$

■

3.3 Finite Element Spaces, Discrete Elliptic Operators and Stiffness Matrices

Corresponding to the triangulations $\mathcal{T}_k$, the finite element spaces $\mathcal{M}_k$ will be defined by

$$\mathcal{M}_k = \{v \in H_0^1(\Omega_k) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall \tau \in \mathcal{T}_k\}.$$

Let $\{x_i^k\}$ be the set of all interior nodes of $\mathcal{T}_k$ and $\{\phi_1, \ldots, \phi_{n_k}\}$ be the standard piecewise linear nodal basis functions such that ϕ_i is equal to one at precisely one node x_i^k and vanishs at all other nodal points. It is clear that ϕ_i is locally supported. In fact

$$\text{supp } \phi_i = \bigcup_{\tau \in \mathcal{T}_k, x_i^k \in \tau} \tau.$$

Associated with each node, we define a local mesh size:

$$h_{k,i} = \frac{1}{2} \text{diam supp } \phi_i.$$

We define a discrete inner product on $\mathcal{M}_k$ by [1]

$$(u, v)_k = h_k^2 \sum_{i=1}^{n_k} h_{k,i}^{d-2} (uv)(x_i^k), \quad u, v \in \mathcal{M}_k, \tag{3.7}$$

and the induced norm is denoted by $\|\cdot\|_k$. It is straightforward to see that

$$\|u\|_k{}^2 \asymp h_k^2 \sum_{\tau \in \mathcal{T}_k} h_\tau^{-2} \|u\|_{L^2(\tau)}^2. \tag{3.8}$$

If $\{\mathcal{T}_k, k \in \mathcal{I}\}$ is quasiuniform, $(\cdot, \cdot)_k$ may be alternatively defined by

$$(u, v)_k = h_k^d \sum_{i=1}^{n_k} (uv)(x_i^k), \quad u, v \in \mathcal{M}_k. \tag{3.9}$$

[1]The factor h_k^2 in this definition is not essential and hence can be omitted. We keep it here simply for the convenience of the analysis because for quasiuniform meshes the discrete norm $\|\cdot\|_k$ defined this way is equivalent to the ordinary L^2-norm in $\mathcal{M}_k$.

With respect to the discrete product $(\cdot,\cdot)_k$ for each k, we define an operator $A_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ by

$$(A_k u, v)_k = A(u,v), \quad u, v \in \mathcal{M}_k. \tag{3.10}$$

where $A(\cdot,\cdot)$ is given by (3.3). This operator will be used later on, and may regarded as a discretized elliptic operator $\mathcal{L}$ given in (3.1).

The following matrix is often called the stiffness matrix:

$$A^k \stackrel{\text{def}}{=} (A(\bar{\phi}_i, \bar{\phi}_j))_{n_k \times n_k},$$

where $\{\bar{\phi}_i\} = \{h_{k,i}^{\frac{2-d}{2}} \phi_i\}$ are the scaled basis functions.

The reasons for using scaled nodal basis to define the stiffness matrix will become clear when we discuss its condition number in Section 3.13.

The stiffness matrix A^k is the one that one has to use in actual programming on computers. However it will play no roll in later analysis or even in the formulation of our multigrid algorithms. Instead, the operator A_k will be used, since we found this more convenient in our analysis. As one may expect, A^k and A_k are closely related. In order to give an explicit formulation, we first need to relate the vector function space $\mathcal{M}_k$ and the Euclidean space $\mathbb{R}^{n_k}$. These two spaces are actually isomorphic and a rather natural isomorphism $\Gamma_k : \mathcal{M}_k \mapsto \mathbb{R}^{n_k}$ can be defined by

$$(\Gamma_k u)_i \stackrel{\text{def}}{=} h_{k,i}^{\frac{d-2}{2}} u(x_i^k).$$

It is easy to see that Γ_k is indeed an isomorphism and furthermore it gives rise the following identity:

$$u = \sum_{i=1}^{n_k} (\Gamma_k u)_i \bar{\phi}_i.$$

In terms of Γ_k, we can then formulate the relationship between A^k and A_k as follows:

Proposition 3.1 *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{\Gamma_k} & \mathbb{R}^{n_k} \\ A_k \big\downarrow & & \big\downarrow h_k^{-2} A_k \\ \mathcal{M}_k & \xrightarrow{\Gamma_k} & \mathbb{R}^{n_k} \end{array}$$

Namely,

$$h_k^{-2} A^k \Gamma_k = \Gamma_k A_k, \quad \textit{or} \quad h_k^{-2} A^k = \Gamma_k A_k \Gamma_k^{-1}.$$

This proposition somehow identifies the operator A_k and the matrix A^k. Because of the above relationship, one may say that A_k and $h_k^{-2} A^k$ are *similar* to each other. As a direct consequence, A_k and $h_k^{-2} A^k$ have analogous spectral properties. In particular we have

$$\kappa(A_k) = \kappa(A^k),$$

where κ is the condition number.

Proof. For any $u \in \mathcal{M}_k$, it follows from definition-tracing that

$$\begin{aligned} (\Gamma_k A_k u)_i &= h_{k,i}^{\frac{d-2}{2}} (A_k u)(x_i) = h_k^{-2} (A_k u, \phi_i)_k = h_k^{-2} A(u, \phi_i) \\ &= h_k^{-2} \sum_j A(\bar{\phi}_i, \bar{\phi}_j)(\Gamma_k u)_j = h_k^{-2} (A^k \Gamma_k u)_i, \end{aligned}$$

as desired. ■

3.4 Inverse Inequalities

Inverse properties are important in the theory of finite elements, because they allow one to relate various Sobolev norms of finite element functions. In the following we will include a few useful inverse inequalities. In particular, we give a proof of an inverse inequality in the fractional norm. To the author's knowledge no proof of this result is available in the literature; an alternative proof may be found in the joint paper by the present author, Bramble and Pasciak [25].

3.4.1 Local Inverse Inequalities

The ordinary inverse inequalities are based on the the local inequalities on each element of the triangulation as follows:

Proposition 3.2 *Assume that* $\{\mathcal{T}_k, k \in \mathcal{I}\}$ *is satisfies (***A3.1***), then for any* $p \geq d, k \in \mathcal{I}, \tau \in \mathcal{T}_k$ *and* $u \in \mathcal{M}_k$,

$$|u|_{H^1(\tau)} \lesssim h_\tau^{-1} \|u\|_{L^2(\tau)},$$

and

$$\|u\|_{W^{m,\infty}(\tau)} \lesssim h_\tau^{-(m+\frac{d}{p})} \|u\|_{L^p(\tau)}, \quad m = 0, 1.$$

The idea of the proof is to map τ to a fixed unit size reference element and then utilize the fact that any two norms on a fixed finite dimensional space are equivalent. For details we refer to [36].

3.4.2 Global Inverse Inequalities

The following global result can be derived immediately from Proposition 3.2.

Proposition 3.3 *For any* $p \geq d, k \in \mathcal{I}, \tau \in \mathcal{T}_k$ *and* $u \in \mathcal{M}_k$,

$$|u|_{H^1(\Omega)} \lesssim \underline{h}_k^{-1} \|u\|_{L^2(\Omega)}, \tag{3.11}$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} \lesssim \underline{h}_k^{-(m+\frac{d}{p})} \|u\|_{L^p(\Omega)}, \quad m = 0, 1. \tag{3.12}$$

In particular, if $\{\mathcal{T}_k\}$ *is quasiuniform, the above two inequalities hold with* h_k *in place of* $\underline{h}_k$

Taking $p = |\log \underline{h}_k|$, we immediately get

Corollary 3.1 $\|\cdot\|_{L^\infty(\Omega)}$ *and* $\|\cdot\|_{L^{|\log \underline{h}_k|}(\Omega)}$ *are uniformly equivalent norms on* $\mathcal{M}_k$.

This result seems interesting since it indicates that in the finite element spaces certain L^p norms actually dominate the L^∞ norm uniformly, which seems to be first observed by Chen [35].

3.4.3 Fractional Inverse Inequalities

Next we are going to show an inverse inequality in a fractional Sobolev norm. We start with a preliminary result on more general polynomials.

Lemma 3.2 *Let m be a nonnegative integer and*

$$S_{k,m} \stackrel{\text{def}}{=} \{v \in L^2(\Omega) : v|_\tau \in \mathcal{P}_m(\tau), \quad \forall \tau \in \mathcal{T}_k\}.$$

then

$$\|v\|_{H^\beta(\Omega)} \lesssim h_k^{-\beta} \|v\|_{L^2(\Omega)}, \quad \forall v \in S_{k,m}, \tag{3.13}$$

for $\beta \in [0, \frac{1}{2})$.

Proof. By (2.1), one has

$$\|u\|^2_{H^\beta(\Omega)} = \|u\|^2_{L^2(\Omega)} + |v|^2_{H^\beta(\Omega)}$$

where

$$|v|^2_{H^\beta(\Omega)} = \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{d+2\beta}} dxdy.$$

Evidently the above integration can be written as a summation of the following three integrals:

$$I_1 \stackrel{\text{def}}{=} \sum_{\tau \cap \tau' = \emptyset} \int_\tau \int_{\tau'} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2\beta}} dxdy,$$

$$I_2 \stackrel{\text{def}}{=} \sum_{\tau \cap \tau' \neq \emptyset, \tau \neq \tau'} \int_\tau \int_{\tau'} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2\beta}} dxdy,$$

$$I_3 \stackrel{\text{def}}{=} \sum_{\tau} \int_\tau \int_{\tau} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2\beta}} dxdy.$$

Because of the symmetric position of τ and τ', we have

$$\begin{aligned} I_1 &\le 2\sum_{\tau\cap\tau'=\emptyset}\int_\tau\int_{\tau'}\frac{|v(x)|^2}{|x-y|^{d+2\beta}}dxdy \\ &\lesssim \sum_\tau\int_\tau|v(x)|^2dx\int_{|y-x|\ge ch}\frac{1}{|x-y|^{d+2\beta}}dy \\ &\lesssim h^{-2\beta}\|v\|_{L^2}. \end{aligned}$$

Now take $\tau\neq\tau', \tau\cap\tau'\neq\emptyset$, for $x\in\tau$, and define

$$d_x = dist\{x,\tau'\}.$$

Then

$$\sum_{\tau\cap\tau'\neq\emptyset,\tau\neq\tau'}\int_{\tau'}\frac{dy}{|x-y|^{d+2\beta}}\le 2\pi\int_{d_x}^\infty\frac{r^{d-1}dr}{r^{d+2\beta}}\lesssim d_x^{-2\beta}.$$

Since $\beta<\frac{1}{2}$, it is easy to check that

$$\int_\tau d_x^{-2\beta}\lesssim h^{2-2\beta}.$$

Consequently,

$$I_2\lesssim h^{2-2\beta}\sum_\tau\|v\|_{0,\infty,\tau}^2\lesssim h^{-2\beta}\|v\|_{L^2(\Omega)}^2.$$

The estimate for I_3 is easy, we have

$$\begin{aligned} I_3 &\le \sum_\tau|v|_{1,\infty,\tau}^2\int_\tau\int_\tau\frac{|x-y|^2}{|x-y|^{d+2\beta}}dxdy \\ &\lesssim h^{4-2\beta}\sum_\tau|v|_{1,\infty,\tau}^2\lesssim h^{-2\beta}\|v\|_{L^2(\Omega)}. \end{aligned}$$

The desired result then follows. ■

As a direct consequence of the above lemma, we have the inverse inequality for our finite element functions in fractional Sobolev norms as follows:

Theorem 3.1 *If $\beta \in [0, \frac{1}{2})$, then*

$$\|v\|_{H^{1+\beta}(\Omega)} \lesssim h_k^{-\beta} \|v\|_{H^1(\Omega)}, \quad \forall v \in \mathcal{M}_k.$$

Above all, the above theorem shows that a continuous piecewise polynomial actually belongs to $H^{1+\beta}(\Omega)$ for $\beta \in [0, \frac{1}{2})$, which is not obvious but a known fact.

3.5 Quasi-uniformity and Mass Matrix

Quasiuniformity is an important property of a family of triangulations. One has to be very careful in situations where this assumption is violated. Without quasiuniformity, many nice properties simply break down, as we will see in the following.

Given $u \in \mathcal{M}_k$, we have

$$\begin{aligned} \|u\|^2_{L^2(\Omega)} &= \sum_{\tau \in \mathcal{T}_k} \|u\|^2_{L^2(\tau)} \\ &\asymp \sum_{\tau \in \mathcal{T}_k} h_\tau^d \sum_{x \in \tau \cap \mathcal{N}_k} |u(x)|^2 \qquad (3.14) \\ &\asymp \sum_{i=1}^{n_k} h_{k,i}^d |u(x_i^k)|^2. \qquad (3.15) \end{aligned}$$

By the *mass matrix* we mean that

$$M^k \stackrel{\text{def}}{=} ((\bar{\phi}_i, \bar{\phi}_j))_{n_k \times n_k}.$$

For $\xi \in \mathbb{R}^{n_k}$, we have

$$\begin{aligned} < M^k \xi, \xi > &= \sum_{i,j=1}^{n_k} (\bar{\phi}_i, \bar{\phi}_j) \xi_i \xi_j = \|\sum_{j=1}^{n_k} \xi_j \bar{\phi}_j\|^2_{L^2(\Omega)} \\ &\asymp \sum_{i=1}^{n_k} h_{k,i}^d |\sum_{j=1}^{n_k} \xi_j \bar{\phi}_j(x_i^k)|^2 \asymp \sum_{i=1}^{n_k} h_{k,i}^2 \xi_i^2. \end{aligned}$$

Consequently we have shown that

Proposition 3.4

$$< M^k\xi, \xi > \asymp < M_0^k\xi, \xi >, \quad \forall \xi \in \mathbb{R}^{n_k},$$

where

$$M_0^k = diag\,(h_{k,i}^2).$$

Proposition 3.5 *The following are equivalent:*

1. $\{\mathcal{T}_k : k \in \mathcal{I}\}$ *is quasiuniform.*

2. *For all* $u \in \mathcal{M}_k$

$$\|u\| \asymp \|u\|_k. \tag{3.16}$$

3.

$$< M^k\mu, \mu > \asymp h_k^2|\mu|^2, \quad \forall \mu \in \mathbb{R}^{n_k}. \tag{3.17}$$

4. *The following inverse inequality holds:*

$$|u|_{H^1(\Omega)} \leq Ch_k^{-1}\|u\|_{L^2(\Omega)}, \quad \forall u \in \mathcal{M}_k. \tag{3.18}$$

Proof. The equivalency of 1), 2) and 3) follows directly from above proposition. It was known before that 1) implies 4). To see the contrary, we take $u = \phi_i$ in (3.18) and obtain that $h_k \lesssim h_{k,i}$ $\forall i$, which implies 1). This completes the proof. ■

3.6 Some Estimates for the Interpolation Operator

In this section, we will provide some estimates for the nodal value interpolation operator $I_k : C(\bar{\Omega}) \mapsto \mathcal{M}_k$ defined by

$$(I_k u)(x_i^k) = u(x_i^k), \quad \forall x_i^k \in \mathcal{N}_k.$$

The first result to be included is well-known and can be obtained by the Bramble-Hilbert Lemma.

Theorem 3.2 *Assume $d \le 3$ and $\{\mathcal{T}_k, k \in \mathcal{I}\}$ is quasiuniform, then*

$$\|(I - I_k)v\|_{L^2(\Omega)} + h_k\|(I - I_k)v\|_{H^1(\Omega)} \lesssim h_k^2\|v\|_{H^2(\Omega)}, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

The assumption that $d \le 3$ in above theorem is to guarantee $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ which is necessary to control the interpolation operator. The following result however relaxes this restriction in some special circumstances.

Lemma 3.3 *Assume $\{\mathcal{T}_k, k \in \mathcal{I}\}$ satisfies (**A3.1**) and $l \ge 1$ is given. Then we have*

$$\|(I - I_k)v\|_{L^p(\tau)} \lesssim h_\tau\|\nabla v\|_{L^p(\tau)}, \quad \forall v \in \mathcal{P}_l(\tau),\ \forall 1 \le p \le \infty, \forall \tau \in \mathcal{T}_k.$$

The estimate provided in above lemma does not in general hold for $v \in W^{1,p}(\tau)$ when the imbedding $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ does not hold (e.g. $p \le d$).

Proof. Let $\hat{\tau}$ be the standard reference element, for any $\tau \in \mathcal{M}_k$, we have an affine diffeomorphism:

$$F_\tau : \hat{\tau} \mapsto \tau.$$

For any function $v \in L^2(\tau)$, we adopt the following standard notation:

$$\hat{v}(\hat{x}) = v(F(\hat{x})), \quad \hat{x} \in \hat{\tau}.$$

Due to (**A3.1**), we can show that (cf.[36])

$$\|(I - I_k)v\|_{L^p(\tau)} \lesssim h_\tau^{\frac{d}{p}}\|(\hat{I} - \hat{I}_k)\hat{v}\|_{L^p(\hat{\tau})}. \tag{3.19}$$

Since $\mathcal{P}_l(\hat{\tau})$ is a fixed finite dimensional space, we get from (2.13) that

$$\|(\hat{I} - \hat{I}_k)\hat{v}\|_{L^p(\hat{\tau})} \lesssim \|\hat{v}\|_{W^{1,p}(\hat{\tau})}.$$

Replacing $\hat{v}$ by $\hat{v} + \hat{q}$ for $\hat{q} \in \mathcal{P}_1(\hat{\tau})$, we get

$$\|(\hat{I} - \hat{I}_k)\hat{v}\|_{L^p(\hat{\tau})} \lesssim \inf_{\hat{q} \in \mathcal{P}_1(\hat{\tau})} \|\hat{v} + \hat{q}\|_{W^{1,p}(\hat{\tau})}$$

$$\lesssim \|\nabla\hat{v}\|_{L^p(\hat{\tau})} \lesssim h_\tau^{1-\frac{d}{p}}\|\nabla v\|_{L^p(\tau)}.$$

The desired result then follows by combining the above estimate with (3.19). ■

Theorem 3.3 *Assume* $\{\mathcal{T}_k, k \in \mathcal{I}\}$ *is quasiuniform and* $\beta, \delta \in [0, 1)$ *are given. Then we have*

$$\|(I - I_k)v\|_{H^{1-\beta}(\Omega)} \lesssim h_k^{\beta+\delta}\|v\|_{H^{1+\delta}(\Omega)}, \quad \forall v \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega).$$

Proof. By the interpolation property of Sobolev space, it suffices to show that

$$\|(I - I_k)v\|_{L^2(\Omega)} \lesssim h_k^{1+\delta}\|v\|_{H^{1+\delta}(\Omega)}, \quad \forall v \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega), \tag{3.20}$$

and

$$\|(I - I_k)v\|_{H^1(\Omega)} \lesssim h_k^{\delta}\|v\|_{H^{1+\delta}(\Omega)}, \quad \forall v \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega). \tag{3.21}$$

Proofs of (3.20) and (3.21) are similar. Hence we will only show (3.20). For this purpose, it suffices to show that

$$\|(I - I_k)v\|_{L^2(\tau)} \lesssim h_k^{\delta}|\nabla v|_{H^\delta(\tau)}, \tag{3.22}$$

since by the integral representation of the fractional norm, we have

$$\sum_{\tau \in \mathcal{T}_k} |\nabla v|_{H^\delta(\tau)}^2 \leq |\nabla v|_{H^\delta(\Omega)}^2.$$

The idea to show (3.22) is again to use the Bramble–Hilbert technique. As before let $\hat{\tau}$ be the reference element, by changing variable, we have

$$\|(I - I_k)v\|_{L^2(\tau)} \leq h_k^{\frac{d}{2}}\|(\hat{I} - \hat{I}_k)\hat{v}\|_{L^2(\hat{\tau})}.$$

Since $H^{1+\delta}(\hat{\tau}) \hookrightarrow C(\hat{\tau})$ and $\hat{I}_k$ is invariant on linears, we get

$$\|(\hat{I} - \hat{I}_k)\hat{v}\|_{L^2(\hat{\tau})} \lesssim \inf_{\hat{q} \in \mathcal{P}_1(\hat{\tau})} \|\hat{v} + \hat{q}\|_{H^{1+\delta}(\hat{\tau})}.$$

In virtue of (2.13) in Theorem (2.3), we have

$$\inf_{\hat{q}\in\mathcal{P}_1(\hat{\tau})} \|\hat{v}+\hat{q}\|_{H^{1+\delta}(\hat{\tau})} \lesssim |\nabla\hat{v}|_{H^\delta(\hat{\tau})}.$$

Mapping $\hat{\tau}$ back to τ, it is elementary to check that

$$|\nabla\hat{v}|_{H^\delta(\hat{\tau})} \lesssim h_k^{1+\delta-\frac{d}{2}} |\nabla v|_{H^\delta(\tau)}.$$

The desired result then follows. ■

3.7 Discrete Sobolev Inequalities

The aim of this section is to prove the following discrete imbedding theorem for general nonquasiuniform meshes.

Theorem 3.4 *Assume* $\{\mathcal{T}_k : k\in\mathcal{I}\}$ *satisfy (***A3.1***), then for any* $v\in\mathcal{M}_k$*,*

$$\|v\|_{C(\bar{\Omega})} \lesssim |\log \underline{h}_k|^{1-\frac{1}{d}} \|v\|_{d/p,p,\Omega}, \qquad p\geq 1, \tag{3.23}$$

in particular,

$$\|v\|_{C(\bar{\Omega})} \lesssim |\log \underline{h}_k|^{1-\frac{1}{d}} \|v\|_{1,d,\Omega}. \tag{3.24}$$

For a special case $d=2$, this type of inequality has been mentioned by many authors. cf. [97], [23].

Proof. Applying Corollary 2.1 in Chapter 2 (for $q=\infty, s=1$) gives,

$$\|v\|_{C(\bar{\Omega})} \lesssim |\log \underline{h}_k|^{1-\frac{1}{d}} \|v\|_{W^{d/p,p}(\Omega)} + \epsilon^\lambda \|v\|_{W^{1,\infty}(\Omega)}. \tag{3.25}$$

Using the inverse property (3.12), we find that

$$\|v\|_{1,\infty,\Omega} \lesssim \underline{h}_k^{-\alpha} \|v\|_{W^{\frac{d}{p},p}(\Omega)}, \tag{3.26}$$

for some $\alpha>0$ depending on d,p. The proof is therefore completed by taking $\epsilon=\underline{h}_k^\alpha$. ■

3.8 L^2 Projections and the Simultaneous Approximation Property

For each $k \in \mathcal{I}$, the L^2 projection $Q_k : L^2(\Omega) \mapsto \mathcal{M}_k$ is defined by

$$(Q_k u, v) = (u, v), \quad \forall u \in L^2(\Omega), v \in \mathcal{M}_k.$$

The aim of this section is to establish an error estimate for Q_k on H^1 in the both the L^2 and H^1 norms, namely

$$\|u - Q_k u\|_{L^2(\Omega)} + h_k \|u - Q_k u\|_{H^1(\Omega)} \lesssim h_k \|u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Using the triangle and the inverse inequality, it is not difficult to show that the above estimate is equivalent to the following so-called simultaneous approximation property:

$$\inf_{\chi \in \mathcal{M}_k} \{\|u - \chi\|_{L^2(\Omega)} + h_k \|u - \chi\|_{H^1(\Omega)}\} \lesssim h_k \|u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \qquad (3.27)$$

This property has been assumed in some papers on finite elements. But it seems that no proof is available in the literature. This estimate is actually true in any number of dimensions. Bramble showed to the author that an application of the results in Hilbert [51] would give rise to a rather simple proof. Another proof may be obtained by using the standard average smoothing operator and Zygmund-Littlewood universal extension theorem, which was shown to the author by Schatz and Wahlbin. Recently R. Scott and S. Zhang [81] have constructed a kind of interpolation operator for nonsmooth functions that can also be used to give a proof of this result. In order to avoid some technical details, we have decided to take a more elementary approach here to establish the result in special cases where $d \leq 3$, which is sufficient in most applications.

Error Estimates

Theorem 3.5 *For* $\beta = 0, or\, 1,$ *and* $u \in H^{1+\beta}(\Omega) \cap H_0^1(\Omega)$

$$\|u - Q_k u\|_{L^2(\Omega)} \lesssim h_k^{1+\beta} \|u\|_{H^{1+\beta}(\Omega)} \tag{3.28}$$

$$\|u - Q_k u\|_{H^1(\Omega)} \lesssim h_k^{\beta} \|u\|_{H^{1+\beta}(\Omega)}. \tag{3.29}$$

Proof. In the cases $d \leq 3$, we have the following Sobolev imbedding:

$$H^2(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Hence by using the nodal value interpolant I_k and applying Theorem 3.2, we can get

$$\|u - Q_k u\|_{L^2(\Omega)} \leq \|u - I_k u\|_{L^2(\Omega)} \lesssim h_k^2 \|u\|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

which is just (3.28) with $\beta = 1$. On the other hand

$$\|u - Q_k u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega).$$

Therefore (3.28) with $\beta = 0$ follows by applying interpolation to above two estimates. The proof of (3.29) is similar to (3.28) by considering H^1-norm in place of L^2-norm and using the stability of Q_k in the H^1-norm which is provided in Theorem 3.6 below. ■

Stability

The stability of the L^2 projection in the H^1 norm was perhaps first proven by Bank and Dupont in [10]. In their proof, it was assumed that the regularity condition (3.5) holds for $\alpha = 1$. As we pointed our earlier, it can be directly obtained assuming the simultaneous approximation property (3.27) which is actually the approach that Mandel, McCormick and Bank take in [66]. A more general discussion of this problem can be found in Crouzeix and Thomee [38]. Our treatment here is not to take the simultaneous approximation property for granted, instead we will prove the stability by a local argument and then we use the stability result to prove the simultaneous approximation property.

Theorem 3.6 *If* $0 \le \beta \le 1$, $\beta \ne \frac{1}{2}$, *then for all* $u \in H^\beta(\Omega)$ $(\beta < \frac{1}{2})$ *or* $H_0^\beta(\Omega)$ $(\beta > \frac{1}{2})$

$$\|Q_k u\|_{H^\beta(\Omega)} \lesssim \|u\|_{H^\beta(\Omega)}. \tag{3.30}$$

By the interpolation technique, it suffices to show (3.30) for $\beta = 1$. The main ingredient of the proof is the local L^2 projection $Q_\tau : L^2(\tau) \mapsto \mathcal{P}_1(\tau)$, for any given $\tau \in \mathcal{T}_k$ defined by

$$(Q_\tau u, \phi)_{L^2(\tau)} = (u, \phi)_{L^2(\tau)}, \quad \forall u \in L^2(\tau),\ \phi \in \mathcal{P}_1(\tau).$$

Let $\hat{\tau}$ be the standard reference element and $\hat{}$ is as defined as in the proof of Lemma 3.3. If $Q_{\hat{\tau}}$ is defined similarly, it is then straightforward to verify that

$$\widehat{Q_\tau u} = Q_{\hat{\tau}} \hat{u}. \tag{3.31}$$

Lemma 3.4

$$|Q_\tau u|_{H^1(\tau)} \lesssim |u|_{H^1(\tau)}, \quad \forall u \in H^1(\tau), \tag{3.32}$$

and

$$\|u - Q_\tau u\|_{L^2(\tau)} \lesssim h_k |u|_{H^1(\tau)}, \quad \forall u \in H^1(\tau). \tag{3.33}$$

Proof. It follows from (3.31) that (3.32) is equivalent to

$$|Q_{\hat{\tau}} \hat{u}|_{H^1(\hat{\tau})} \lesssim |\hat{u}|_{H^1(\hat{\tau})}, \quad \forall \hat{u} \in H^1(\hat{\tau}). \tag{3.34}$$

Since all the norms on $\mathcal{P}_1(\hat{\tau})$ are equivalent, we have

$$|Q_{\hat{\tau}} \hat{u}|_{H^1(\hat{\tau})} \lesssim \|Q_{\hat{\tau}} \hat{u}\|_{L^2(\hat{\tau})} \le \|\hat{u}\|_{L^2(\hat{\tau})} \lesssim \|\hat{u}\|_{H^1(\hat{\tau})},$$

which, since $Q_{\hat{\tau}} \hat{c} = \hat{c}$ for any $\hat{c} \in \mathbb{R}^1$, implies that

$$|Q_{\hat{\tau}} \hat{u}|_{H^1(\hat{\tau})} \lesssim \inf_{\hat{c} \in \mathbb{R}^1} \|\hat{u} + \hat{c}\|_{H^1(\hat{\tau})} \lesssim |\hat{u}|_{H^1(\hat{\tau})}.$$

This proves (3.34) and hence (3.32).

Now we come to the proof of (3.33). By changing variables and using (3.31), we get

$$\begin{aligned}\|u - Q_\tau u\|_{L^2(\tau)} &\lesssim h_k^{d/2}\|\hat{u} - \hat{Q}_{\hat{\tau}}\hat{u}\|_{L^2(\hat{\tau})}\\ &\lesssim h_k^{d/2}\inf_{\hat{c}\in\mathbb{R}^1}\|\hat{u}+\hat{c}\|_{H^1(\hat{\tau})} \lesssim h_k^{d/2}|\hat{u}|_{H^1(\hat{\tau})}\\ &\lesssim h_k^{d/2}h_k^{1-d/2}|u|_{H^1(\tau)} \lesssim h_k|u|_{H^1(\tau)}.\end{aligned}$$

This completes the proof. ■

Proof of Theorem 3.6:

It follows from the inverse inequality, Lemma 3.5, and Lemma 3.4, that

$$\begin{aligned}|Q_k u|^2_{H^1(\Omega)} &= \sum_{\tau\in\mathcal{T}_k}|Q_k u|^2_{H^1(\tau)}\\ &\leq 2\sum_{\tau\in\mathcal{T}_k}\{|Q_k u - Q_\tau u|^2_{H^1(\tau)} + |Q_\tau u|^2_{H^1(\tau)}\}\\ &\lesssim \sum_{\tau\in\mathcal{T}_k}\{h_k^{-2}|Q_k u - Q_\tau u|^2_{L^2(\tau)} + |u|^2_{H^1(\tau)}\}\\ &\lesssim \sum_{\tau\in\mathcal{T}_k}\{h_k^{-2}\|u - Q_\tau u\|^2_{L^2(\tau)} + |u|^2_{H^1(\tau)}\} + h_k^{-2}\|u - Q_k u\|^2_{L^2(\Omega)}\\ &\lesssim |u|_{H^1(\Omega)}.\end{aligned}$$

The desired result then follows. ■

Simultaneous Approximation Properties

Proposition 3.6 *For any $u \in H^{1+\beta}(\Omega)\cap H_0^1(\Omega)$ ($\beta = 0$ or 1), there exists $v_k \in \mathcal{M}_k$ such that*

$$\|u - v_k\|_{L^2(\Omega)} + h_k\|u - v_k\|_{H^1(\Omega)} \lesssim h_k^{1+\beta}\|u\|_{H^{1+\beta}(\Omega)}. \tag{3.35}$$

Proof. Take $v_k = Q_k u$. ■

The above simultaneous approximation property holds for more general boundary conditions. For example, if $\Gamma_0 \subset \partial\Omega$ is a measurable subset, then we have

Proposition 3.7 *For any $u \in H^1_{\Gamma_0}(\Omega)$, there exists $v_k \in \mathcal{M}_k \cap H^1_{\Gamma_0}(\Omega)$ such that* (3.35) *holds.*

3.9 Weighted L^2 Projection

The motivation of this section is to study the convergence property of the multigrid method applied to interface problems with strongly discontinuous coefficients, applications can be found in Section 10.4.

Assume the domain Ω admits the following decomposition:

$$\bar{\Omega} = \bigcup_{i=1}^{J} \bar{\Omega}_i, \tag{3.36}$$

where Ω_i are mutually disjoint.

Let Γ denote the set of interfaces, namely

$$\Gamma = \cup_{i=1}^{J} \partial\Omega_i \setminus \partial\Omega.$$

For simplicity, we assume that Γ consists only of segments ($d = 2$) or plane polygons ($d = 3$). In other words, no part of any $\partial\Omega_i$ away from $\partial\Omega$ is curved.

Given a set of positive constants $\{\omega_i\}_{i=1}^{J}$, we introduce the following weighted inner products:

$$(u, v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i (u, v)_{L^2(\Omega_i)}, \tag{3.37}$$

and

$$(u, v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i (\nabla u, \nabla v)_{L^2(\Omega_i)}, \tag{3.38}$$

with the induced norms denoted by $\|\cdot\|_{L^2_\omega(\Omega)}$ and $|\cdot|_{H^1_\omega(\Omega)}$, respectively.

As is done in Section 3.5.1, we assume that Ω is triangulated by a nested sequence of quasiuniform meshes $\{\mathcal{T}_k : k = 1, \ldots, j\}$. An obvious additional assumption we need here is that these triangulations will be lined up with the subdomains Ω_i's. Namely the restriction of each $\mathcal{T}_k$ on each Ω_i is also a triangulation of Ω_i itself.

The weighted L^2 projection $Q_k^\omega : L^2(\Omega) \mapsto \mathcal{M}_k$ is defined by

$$(Q_k^\omega u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)}, \qquad \forall u \in L^2(\Omega), v \in \mathcal{M}_k. \tag{3.39}$$

We will derive error estimates for Q_k^ω of the following type:

$$\|(I - Q_k^\omega)u\|_{L^2_\omega(\Omega)} \leq Ch_k |u|_{H^1_\omega(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

The point here is that we require that the constant C appearing in the above estimate does not depend on the weights $\{\omega_i\}$. Again we will use the notation "$\lesssim$" in place of "$\leq C$", where C is in particular independent of ω_i's.

We introduce a weighted inner product on $L^2(\Gamma)$ as follows

$$(u, v)_{L^2_\omega(\Gamma)} = \sum_{i=1}^{J} \int_{\partial\Omega_i \backslash \partial\Omega} \omega_i uv dx.$$

Denoting $\mathcal{M}_k(\Gamma) \stackrel{\text{def}}{=} \{v|_\Gamma : \ v \in \mathcal{M}_k\}$, let $P_\Gamma^k : L^2(\Gamma) \mapsto \mathcal{M}_k(\Gamma)$ be the orthogonal projection with respect to $(\cdot, \cdot)_{L^2_\omega(\Gamma)}$

The following lemma shows that the estimate we need can be reduced to the estimates on interfaces.

Lemma 3.5 *For all* $u \in H_0^1(\Omega)$

$$\|(I - Q_k^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h_k |u|_{H^1_\omega(\Omega)} + h_k^{1/2} \|u - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}.$$

Proof. On each domain Ω_i, by simultaneous approximation property, there exists a $w_i \in \mathcal{M}_k(\Omega_i)$ such that

$$\|u - w_i\|^2_{L^2(\Omega_i)} + h_k^2 \|u - w_i\|^2_{H^1(\Omega_i)} \lesssim h_k^2 \|u\|^2_{H^1(\Omega_i)}.$$

Let $w \in \mathcal{M}_k$ be such that

$$w = \begin{cases} w_i, & \text{at the nodes in } \Omega_i; \\ P_\Gamma^k u, & \text{on } \Gamma. \end{cases}$$

Therefore

$$\begin{aligned}
\|u - w\|^2_{L^2_\omega(\Omega)} &\lesssim \sum_{i=1}^J \omega_i \|u - w_i\|^2_{L^2(\Omega_i)} + h_k^2 \sum_{i=1}^J \omega_i \sum_{p \in \Gamma_i} |(w - P_\Gamma^k u)(p)|^2 \\
&\lesssim \sum_{i=1}^J \omega_i \|u - w_i\|^2_{L^2(\Omega_i)} + h_k \|u - P_\Gamma^k u\|^2_{L^2_\omega(\Gamma)} + h_k \|u - w\|^2_{L^2_\omega(\Gamma)} \\
&\lesssim \sum_{i=1}^J \omega_i h_k^2 \|u\|^2_{H^1(\Omega_i)} + h_k \|u - P_\Gamma^k u\|^2_{L^2_\omega(\Gamma)} \\
&\lesssim h_k^2 |u|_{H^1_\omega(\Omega)} + h_k \|u - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}.
\end{aligned}$$

The desired result then follows since

$$\|u - Q_k^\omega u\|_{L^2_\omega(\Omega)} \le \|u - w\|_{L^2_\omega(\Omega)}.$$

■

Theorem 3.7 *Assume the decomposition* (3.36) *has no crossing points, namely there is no point on* Γ *that belongs to more than two* $\bar{\Omega}_i$*'s. Then, for all* $u \in H_0^1(\Omega)$

$$\|(I - Q_k^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h_k |u|_{H^1_\omega(\Omega)}, \tag{3.40}$$

and

$$|Q_k^\omega u|_{H^1_\omega(\Omega)} \lesssim |u|_{H^1_\omega(\Omega)}. \tag{3.41}$$

Proof. Define a function $\phi \in \mathcal{M}_k(\Gamma)$ by

$$\phi = P^k_{\Gamma_i} u \quad \text{on each } \Gamma_i,$$

where $P^k_{\Gamma_i} : L^2(\Gamma_i) \mapsto \mathcal{M}(\Gamma_i)$ is the orthogonal L^2-projection. By the hypothesis that Γ has no crossing point, ϕ is well defined. Note that on each Γ_i, we have

$$\|u - \phi\|_{L^2(\Gamma_i)} \leq \|u - w_i\|_{L^2(\Gamma_i)}.$$

Due to Lemma 2.2

$$\|u - w_i\|^2_{L^2(\Gamma_i)} \leq h_k^{-1} \|u - w_i\|^2_{L^2(\Omega_i)} + h_k \|u - w_i\|^2_{H^1(\Omega_i)}.$$

Hence,

$$\begin{aligned} h_k \|u - w_i\|^2_{L^2(\Gamma_i)} &\lesssim \|u - w_i\|^2_{L^2(\Omega_i)} + h_k^2 \|u - w_i\|^2_{H^1(\Omega_i)} \\ &\lesssim h_k^2 |u|^2_{H^1(\Omega_i)}. \end{aligned}$$

Consequently,

$$h_k \|u - P^k_\Gamma u\|^2_{L^2_\omega(\Omega)} \lesssim h_k \sum_{i=1}^{J} \omega_i \|u - \phi\|^2_{L^2(\Gamma_i)}$$

$$\lesssim h_k^2 \sum_{i=1}^{J} \omega_i |u|_{H^1(\Omega_i)} = h_k^2 |u|^2_{H^1_\omega(\Omega)}.$$

Applying Lemma 3.5 gives to (3.40).

The proof of (3.41) is identical to that of (3.30) for $\beta = 1$. This completes the proof. ■

Remark 3.1 With a complete different but much more complicated approach, it is possible to entend the result of Theorem (3.7) to more general case (with crossing points). The proof will be reported elsewhere. However, a special two–dimensional result can be found in Section 4.1.3 of the next chapter.

3.10 L^2 Quasi-projection

In this section we will choose the discrete inner product $(\cdot,\cdot)_k$ on $\mathcal{M}_k$ in a special way, namely

$$|(u,v)_k - (u,v)_{L^2(\Omega)}| \lesssim h_k \|u\|_{H^1(\Omega)}\, \|v\|_k, \quad \forall u,v \in \mathcal{M}_k. \tag{3.42}$$

One way to do this is to use a quadrature rule in the ordinary L^2 inner product and define

$$(u,v)_k = \frac{1}{d+1} \sum_{\tau\in\mathcal{T}_k} |\tau| \sum_{x\in\mathcal{N}_k\cap\tau} (uv)(x).$$

We notice that the $k-$norm induced by $(\cdot,\cdot)_k$ defined above satisfies

$$\|v\|_{L^2(\Omega)} \asymp \|v\|_k, \quad \forall v \in \mathcal{M}_k. \tag{3.43}$$

By Lemma 3.3, it is easy to show that this choice of $(\cdot,\cdot)_k$ gives rise to (3.42) in any number of dimensions.

The L^2 *quasi-projection* we shall discuss is an operator $\Pi_k : L^2(\Omega) \mapsto \mathcal{M}_k$ defined by

$$(\Pi_k v, \phi)_k = (v,\phi)_{L^2(\Omega)} \qquad \forall v \in L^2(\Omega), \phi \in \mathcal{M}_k.$$

Notice that unless $(\cdot,\cdot)_k = (\cdot,\cdot)_{L^2(\Omega)}$, Π_k is not a projection, namely $\Pi_k\phi_k \neq \phi_k$ for $\phi_k \in \mathcal{M}_k$ in general.

By definition, we can see that the action of Π_k should be much easier to compute than that of the L^2 projection Q_k but we will show next that properties of Π_k are quite similar to those of Q_k. Hence Π_k may be regarded as something in between Q_k and the nodal value interpolant I_k and it will be a candidate for the so-called *prolongation* operator in the multigrid algorithm when I_k is not likely to work as a prolongation.

Lemma 3.6 *For any* $v \in H_0^1(\Omega)$ *and* $s \in [0,1]$

$$\|\Pi_k v\|_{H^s(\Omega)} \lesssim \|v\|_{H^s(\Omega)}, \tag{3.44}$$

and

$$\|(I - \Pi_k)v\|_{H^s(\Omega)} \lesssim h^{1-s}\|v\|_{H^1(\Omega)}. \tag{3.45}$$

Proof. By definition, for any $\phi \in \mathcal{M}_k$

$$(\Pi_k v, \phi)_k = (v, \phi) \lesssim \|v\|_{L^2(\Omega)}\|\phi\|_k.$$

This, together with (3.43), implies that

$$\|\Pi_k v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\Omega)}. \tag{3.46}$$

Using the L^2- projection Q_k, we get

$$\begin{aligned} & (\Pi_k v - Q_k v, \phi)_k = (v, \phi) - (Q_k v, \phi)_k \\ = \; & (Q_k v, \phi) - (Q_k v, \phi)_k \lesssim h_k \|Q_k v\|_{H^1(\Omega)} \; \|\phi\|_k \\ \lesssim \; & h_k \|v\|_{H^1(\Omega)} \; \|\phi\|_k. \end{aligned}$$

Hence, by (3.43)

$$\|\Pi_k v - Q_k v\|_{L^2(\Omega)} \lesssim h_k \|v\|_{H^1(\Omega)}. \tag{3.47}$$

Since $\|(I - Q_k)v\| \lesssim h_k\|v\|_{H^1(\Omega)}$, the triangle inequality then gives that

$$\|(I - \Pi_k)v\| \lesssim h_k \|v\|_{H^1(\Omega)}. \tag{3.48}$$

It follows from the inverse inequality and (3.47) that

$$\|\Pi_k v - Q_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

Hence

$$\|(I-\Pi_k)v\|_{H^1(\Omega)} \leq \|(I-Q_k)v\|_{H^1(\Omega)} + \|\Pi_k v - Q_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}.$$

Hence

$$\|(I-\Pi_k)v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}. \tag{3.49}$$

or

$$\|\Pi_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)}. \tag{3.50}$$

Interpolating between (3.46) and (3.50) gives (3.44), (3.48) and (3.49) gives (3.45). ■

Remark 3.2 Define $\pi_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ by

$$(\pi_k u, v) = (u, v)_k, \quad \forall u, v \in \mathcal{M}_k.$$

Similar to Lemma 3.6, we can show that, for $\beta = 0$, of 1

$$\|\pi_k v\|_{H^\beta(\Omega)} \lesssim \|v\|_{H^\beta(\Omega)}, \quad \forall v \in \mathcal{M}_k. \tag{3.51}$$

Using the equivalence (3.52), the above inequality can then be extended to $0 < \beta < 1$ by interpolation.

It is straightforward to check that, on $\mathcal{M}_k$, $\Pi_k = \pi_k^{-1}$, therefore combining (3.44) and (3.51), we then get

$$\|\Pi_k v\|_{H^s(\Omega)} \asymp \|v\|_{H^s(\Omega)}, \quad 0 \leq s \leq 1, \quad v \in \mathcal{M}_k.$$

3.11 Galerkin Projections

For each $k \in \mathcal{I}$, the Galerkin projection $P_k : H_0^1(\Omega) \mapsto \mathcal{M}_k$ is defined by

$$A(P_k u, v) = A(u, v), \quad \forall u \in H_0^1(\Omega), v \in \mathcal{M}_k.$$

The following result can be obtained by a standard duality argument with the elliptic regularity estimate (3.5):

Proposition 3.8

$$\|(I-P_k)u\|_{H^{1-\alpha}(\Omega)} \lesssim h_k^\alpha \|u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

where α *is as in* (3.5).

We will have occasion to need the stability results for P_k in certain norms. Its stability in the H^1 norm is trivial, but the following result is not obvious:

Proposition 3.9

$$\|P_k u\|_{H^{1+\beta}(\Omega)} \lesssim \|u\|_{H^{1+\beta}(\Omega)}, \quad \forall u \in H^{1+\beta}(\Omega) \cap H_0^1(\Omega).$$

if $0 \le \beta < \frac{1}{2}$.

The proof to be included below was essentially due to Bramble and Pasciak.

Proof. Setting $\partial_i = \frac{\partial}{\partial x_i}$, we have

$$\|P_k u\|_{H^{1+\beta}(\Omega)} \lesssim \|P_k u\|_{H^1(\Omega)} + \sum_i \|\partial_i P_k u\|_{H^\beta(\Omega)}.$$

Obviously it suffices to estimate the second term. For each i, we have

$$\|\partial_i P_k u\|_{H^\beta(\Omega)} \le \|\partial_i P_k u - Q_k \partial_i u\|_{H^\beta(\Omega)} + \|Q_k \partial_i u\|_{H^\beta(\Omega)}.$$

It is trivial to bound the second term because of (3.30). To see the first term, we use the fractional inverse inequality (3.13) and deduce that

$$\begin{aligned}
&\|\partial_i P_k u - Q_k \partial_i u\|_{H^\beta(\Omega)} \\
&\quad \lesssim h_k^{-\beta} \|\partial_i P_k u - Q_k \partial_i u\|_{L^2(\Omega)} \\
&\quad \lesssim h_k^{-\beta} (\|\partial_i P_k u - \partial_i Q_k u\|_{L^2(\Omega)} + \|\partial_i Q_k u - \partial_i u\|_{L^2(\Omega)} + \|\partial_i u - Q_k \partial_i u\|_{L^2(\Omega)}) \\
&\quad \lesssim h_k^{-\beta} (\|P_k(I-Q_k)u\|_{H^1(\Omega)} + \|(I-Q_k)u\|_{H^1(\Omega)} + \|(I-Q_k)\partial_i u\|_{L^2(\Omega)}) \\
&\quad \lesssim h_k^{-\beta} (\|(I-Q_k)u\|_{H^1(\Omega)} + \|(I-Q_k)\partial_i u\|_{L^2(\Omega)}) \\
&\quad \lesssim \|u\|_{H^{1+\beta}(\Omega)},
\end{aligned}$$

where in the last step we have used Proposition 3.5. The proof is complete. ■

3.12 Equivalence between Discrete and Continuous Fractional Norms

For the elliptic problem (3.1), it is well–known that $\|\cdot\|_{H^s(\Omega)}$, for $0 \le s \le 1$, is equivalent to, say, $\|\mathcal{L}^{\frac{s}{2}}\cdot\|_{L^2(\Omega)}$. Since, as we know, A_k is a discretization of $\mathcal{L}$, the following result is then not surprising:

Lemma 3.7 *Assume* $0 \le s \le 1$, *then*

$$\|A_k^{s/2}u\|_k \asymp \|u\|_{H^s(\Omega)} \quad \forall u \in \mathcal{M}_k. \tag{3.52}$$

If furthermore $(\cdot,\cdot)_k$ *satisfies* (3.42), *then* (3.52) *also holds for* $-1 \le s \le 0$.

Proof. For $0 \le s \le 1$, it was proven by Bank and Dupont in [10] by interpolating the L^2 projection operator and inclusion operator, since for $s = 0$ or 1 the corresponding results are then trivially true. Next we will show the results for negative s by using the result for positive s.

Let $s \in [-1, 0)$, then $s = -\beta$ for $\beta \in (0, 1]$. Hence, for any $v \in \mathcal{M}_k$, we can use the operator π_k defined in the end of last section to deduce that

$$\begin{aligned}
\|A_k^{-\frac{\beta}{2}}v\|_k^2 &= (A_k^{-\beta}v, v)_k = (\pi_k A_k^{-\beta}v, v) \\
&\le \|\pi_k A_k^{-\beta}v\|_{H^\beta(\Omega)}\|v\|_{H^{-\beta}(\Omega)} \lesssim \|A_k^{-\beta}v\|_{H^\beta(\Omega)}\|v\|_{H^{-\beta}(\Omega)} \\
&\lesssim \|A_k^{\frac{\beta}{2}}A_k^{-\beta}v\|_k\|v\|_{H^{-\beta}(\Omega)} = \|A_k^{\frac{-\beta}{2}}v\|_k\|v\|_{H^{-\beta}(\Omega)},
\end{aligned}$$

namely

$$\|A_k^{-\frac{\beta}{2}}v\|_k \lesssim \|v\|_{H^{-\beta}(\Omega)}.$$

On the other hand, for any $\phi \in H^\beta(\Omega)$, using Lemma 3.6 we have

$$\begin{aligned}(v,\phi) &= (v,\Pi_k\phi)_k = (A_k^{-\frac{\beta}{2}}v, A_k^{\frac{\beta}{2}}\Pi_k\phi)_k \\ &\leq \|A_k^{-\frac{\beta}{2}}v\|_k\|\Pi_k\phi\|_{H^\beta(\Omega)} \lesssim \|A_k^{-\frac{\beta}{2}}v\|_k\|\phi\|_{H^\beta(\Omega)},\end{aligned}$$

thus

$$\|v\|_{H^{-\beta}(\Omega)} \lesssim \|A_k^{-\frac{\beta}{2}}v\|_k.$$

This completes the proof. ■

The following norm result shows that the preceding lemma may still hold for some $s > 1$.

Proposition 3.10 *Assume* $(\cdot,\cdot)_k$ *satisfies* (3.42)*. If* $0 \leq \beta < \frac{1}{2}$*, then*

$$\|A_k^{\frac{1+\beta}{2}}u\|_k \lesssim \|u\|_{H^{1+\beta}(\Omega)} \quad \forall u \in \mathcal{M}_k.$$

If furthermore $\beta < \min\{\frac{1}{2},\alpha\}$*, for* α *in* (3.5)*, then*

$$\|u\|_{H^{1+\beta}(\Omega)} \lesssim \|A_k^{\frac{1+\beta}{2}}u\|_k \quad \forall u \in \mathcal{M}_k.$$

Proof. By Lemma 3.7, we have

$$\|A_k^{\frac{1+\beta}{2}}u\|_k \lesssim \|A_ku\|_{H^{\beta-1}(\Omega)}.$$

Taking $\phi \in \mathbf{C}_0^\infty(\Omega)$, and using Lemma 3.6 and Theorem 3.6, we get

$$\begin{aligned}(A_ku,\phi) &= (A_ku,Q_k\phi) = (A_ku,\Pi_kQ_k\phi)_k \\ &= A(u,\Pi_kQ_k\phi) \lesssim \|u\|_{H^{1+\beta}(\Omega)}\|\Pi_kQ_k\phi\|_{H^{1-\beta}(\Omega)} \\ &\lesssim \|u\|_{H^{1+\beta}(\Omega)}\|\phi\|_{H^{1-\beta}(\Omega)}.\end{aligned}$$

This implies that

$$\|A_ku\|_{H^{\beta-1}(\Omega)} \lesssim \|u\|_{H^{1+\beta}(\Omega)}.$$

Hence

$$\|A_k^{\frac{1+\beta}{2}} u\|_k \lesssim \|u\|_{H^{1+\beta}(\Omega)}.$$

This proves the first inequality.

To show the second inequality, we consider the auxiliary problem:

$$\mathcal{L}w = A_k u.$$

Then, it is straightforward to check that

$$u = P_k w.$$

Thus, using Proposition 3.9, (3.5) and Lemma 3.7 we deduce that

$$\begin{aligned} \|u\|_{H^{1+\beta}(\Omega)} &= \|P_k w\|_{H^{1+\beta}(\Omega)} \lesssim \|w\|_{H^{1+\beta}(\Omega)} \\ &\leq \|A_k u\|_{H^{\beta-1}(\Omega)} \lesssim \|A_k^{\frac{\beta-1}{2}} A_k u\|_k \lesssim \|A_k^{\frac{1+\beta}{2}} u\|_k. \end{aligned}$$

The proof is therefore completed. ■

3.13 Conditioning of A_k

It is well-known that the efficiency of most direct or iterative methods for solving a linear system strongly depend on the so-called condition number. Given a SPD operator A on a finite dimensional vector space $\mathcal{M}$, the condition number of A, denoted by $\kappa(A)$, can be defined by

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximum and minimum eigenvalue of A respectively. We note that if μ_0 and μ_1 are constants satisfying

$$\mu_0(v,v) \leq (Av,v) \leq \mu_1(v,v), \quad \forall v \in \mathcal{M}. \tag{3.53}$$

where $(\cdot,\cdot)$ is an inner product on $\mathcal{M}$ with respect to which A is symmetricthen we have

$$\kappa(A) \le \frac{\mu_1}{\mu_0}.$$

Hence, in order to give an upper bound of $\kappa(A)$, we only need to prove an estimate like (3.53).

The discrete elliptic operator A_k we have been studying is known to be ill-conditioned, namely $\kappa(A_k)$ grows very rapidly as the mesh size of the triangulation gets smaller. Some estimates for $\kappa(A_k)$ will be summarized below.

The first estimate for $\kappa(A_k)$ in the case of quasiuniform triangulation was given by Fried [44], who proved that $\kappa(A_k) \lesssim h_k^{-2}$. It was also mentioned (without proof however) by some authors (e.g. Axelsson and Barker [4]) that $\kappa(A_k) \gtrsim h_k^{-2}$. The following theorem includes such results.

Theorem 3.8 *Assume that $\{\mathcal{T}_k, k \in \mathcal{I}\}$ is quasiuniform, then*

$$\lambda_{\max}(A_k) \asymp h_k^{-2},$$

and, in fact

$$\lambda_{\min}(A_k) \asymp 1.$$

Consequently,

$$\kappa(A_k) \asymp h_k^{-2}.$$

Proof. By the definition of A_k, we have

$$\lambda_{\max}(A_k) = \max_{v\in\mathcal{M}_k} \frac{A(v,v)}{\|v\|_k^2},$$

and

$$\lambda_{\min}(A_k) = \min_{v\in\mathcal{M}_k} \frac{A(v,v)}{\|v\|_k^2}.$$

The inverse inequality implies that

$$\lambda_{\max}(A_k) \lesssim h_k^{-2}$$

and it follows from Poincaré inequality and (3.16) that

$$\lambda_{\min}(A_k) \gtrsim 1.$$

Consequently

$$\kappa(A_k) \lesssim h_k^{-2}.$$

Next we choose $v_0 \in \mathcal{M}_k$ such that it is one at the nodes in $\mathcal{N}_k^{\partial}$ ($\stackrel{\text{def}}{=}$ the set of nodes that are adjacent to $\partial\Omega$) and is zero elsewhere. Then

$$\begin{aligned}
\|v_0\|_k{}^2 &= h_k^d \sum_{x \in \mathcal{N}_k^{\partial}} v_0(x)^2 \lesssim h_k^d \sum_{\tau \in \mathcal{T}_k} \sum_{x_1, x_2 \in \tau \cap \mathcal{N}_k} |v_0(x_1) - v_0(x_2)|^2 \\
&= h_k^2 \sum_{\tau \in \mathcal{T}_k} [h_k^d \sum_{x_1, x_2 \in \tau \cap \mathcal{N}_k} |\frac{v_0(x_1) - v_0(x_2)}{h_k}|^2] \\
&\lesssim h_k^2 \sum_{\tau} \int_{\tau} |\nabla v_0|^2 dx \lesssim h_k^2 A(v_0, v_0)
\end{aligned}$$

which proves that $\lambda_{\max}(A_k) \gtrsim h_k^{-2}$, hence $\lambda_{\max}(A_k) \asymp h_k^{-2}$. An upper bound for $\lambda_{\min}(A_k)$ can be obtained similarly by making a special choice of v. This completes the proof. ■

The following result, which is derived by the author from Bank and Scott [12], gives an estimate of $\kappa(A_k)$ when the triangulation is not quasiuniform. We would like to emphasize that for the nonquasiuniform triangulations, the discrete inner product $(\cdot, \cdot)_k$ is defined by (3.7) which is a kind of weighted L^2 inner product and in general its induced norm $\|\cdot\|_k$ is not equivalent to the usual L^2 norm unless the triangulations are quasiuniform.

Theorem 3.9 *Assume that the triangulations $\{\mathcal{T}_k : k \in \mathcal{I}\}$ satisfy (**A3.1**), then we have*

$$\lambda_{d,k}\|u\|_k{}^2 \lesssim (A_k u, u)_k \lesssim h_k^{-2}\|u\|_k{}^2, \quad \forall u \in \mathcal{M}_k. \tag{3.54}$$

where

$$\lambda_{d,k} = \begin{cases} (h_k^d n_k)^{-\frac{2}{d}}, & \text{if } d \geq 3; \\ (h_k^2 n_k)^{-1}(1 + \log \frac{h_k}{\underline{h}_k})^{-1}, & \text{if } d = 2. \end{cases}$$

Consequently

$$\kappa(A_k) = \kappa(A^k) = \begin{cases} n_k^{\frac{2}{d}}, & \text{if } d \geq 3; \\ n_k(1 + \log \frac{h_k}{\underline{h}_k}), & \text{if } d = 2. \end{cases}$$

Proof. In any case, we have

$$\begin{aligned} (A_k u, u)_k &= A(u,u) = \sum_{\tau \in \mathcal{T}_k} A_\tau(u,u) \\ &\lesssim \sum_{\tau \in \mathcal{T}_k} h_\tau^{-2}\|u\|^2_{L^2(\tau)}, \end{aligned}$$

hence, by (4.7) that

$$(A_k u, u)_k \lesssim h_k^{-2}\|u\|_k{}^2.$$

This proves the right hand part of the estimate (3.54). To see its left hand part, we first assume that $d \geq 3$. It follows from Hölder's inequality that

$$\begin{aligned} h_k^{-2}\|u\|_k{}^2 &\lesssim \sum_{\tau \in \mathcal{T}_k} h_\tau^{-2}\|u\|^2_{L^2(\tau)} \lesssim \sum_{\tau \in \mathcal{T}_k} \|u\|^2_{L^{\frac{2d}{d-2}}(\tau)} \\ &\lesssim (\sum_{\tau \in \mathcal{T}_k} 1)^{\frac{2}{d}} \|u\|^2_{L^{\frac{2d}{d-2}}(\Omega)} \lesssim n_k^{\frac{2}{d}}(A_k u, u)_k, \end{aligned}$$

where in the last step we have used Sobolev inequality (2.2).

Now we come to the case where $d = 2$. Let $p > 2$, it follows from (4.7), inverse inequality, Hölder inequality and Sobolev inequality (2.7) that

$$\begin{aligned} h_k^{-2}\|u\|_k{}^2 &\lesssim \sum_{\tau\in\mathcal{T}_k}\|u\|^2_{L^\infty(\tau)} \lesssim \sum_{\tau\in\mathcal{T}_k} h_\tau^{\frac{-4}{p}}\|u\|^2_{L^p(\tau)} \\ &\lesssim \Big(\sum_{\tau\in\mathcal{T}_k} h_\tau^{\frac{-4}{p-2}}\Big)^{\frac{p-2}{p}}\|u\|^2_{L^p(\Omega)} \lesssim p\Big(\sum_{\tau\in\mathcal{T}_k} h_\tau^{\frac{-4}{p-2}}\Big)^{\frac{p-2}{p}} A(u,u), \end{aligned}$$

Notice that $\mathrm{meas}(\Omega) \lesssim \sum_{\tau\in\mathcal{T}_k} h_\tau^2 \lesssim n_k h_k^2$, hence, $n_k^{-1} \lesssim h_k^2$ and

$$\Big(\sum_{\tau\in\mathcal{T}_k} h_\tau^{\frac{-4}{p-2}}\Big)^{\frac{p-2}{p}} \lesssim \underline{h}_k^{\frac{-4}{p}} n_k^{\frac{1-2}{p}} \lesssim \Big(\frac{h_k}{\underline{h}_k}\Big)^{\frac{-4}{p}} n_k.$$

Taking $p = \max(3, \log\frac{h_k}{\underline{h}_k})$ leads to the desired result. ■

Chapter 4

Multilevel Theory of Finite Element Spaces

This is a continuation of the preceding chapter on the theory of finite element approximations. So far, we have studied the family of the triangulations $\{\mathcal{T}_k, k \in \mathcal{I}\}$ without much concern about the index set $\mathcal{I}$. In this chapter, we will take $\mathcal{I} = \{1, 2, \cdots, j\}$ for a given integer j. The corresponding triangulations $\{\mathcal{T}_k, k = 1, \cdots, j\}$ and finite element spaces $\{\mathcal{M}_k : k = 1, \cdots, j\}$ will be the basic ingredients of the multigrid algorithms to be developed. How to set up these triangulations depends on the background of the problem. What we mean by a multilevel theory here in this chapter is that the interaction among spaces on different levels of triangulations is taken account into the analysis.

Basically we will have two different kinds of multilevel spaces, one is related to *nested meshes* while another is related to *nonnested meshes*. Roughly speaking, a sequence of meshes are said to be nested if for each k, $\mathcal{T}_k$ is obtained by refining $\mathcal{T}_{k-1}$, and hence $\mathcal{M}_{k-1}$ is a subspace of $\mathcal{M}_k$, whereas these properties are no longer valid on nonnested meshes. As one may expect, the analysis for nonnested meshes in general is more difficult.

Our main purpose here is to study the relationship among the multilevel spaces $\{\mathcal{M}_k : k = 1, \cdots, j\}$. In order to do that, we need something to link successive spaces. This role is played by the so-called *prolongation* or *restriction*.

The *prolongation* is the operator, denoted by $\mathcal{I}_k$, that transfers functions in $\mathcal{M}_{k-1}$ into $\mathcal{M}_k$. The principles for choosing a prolongation should be first of all its action is relatively easy to compute and secondly it has certain approximation properties. On the nested meshes, the natural inclusion operator is obviously the best candidate, otherwise the interpolant or some other kind of projection operators can be used.

The *restriction* is the operator that takes functions from $\mathcal{M}_k$ into $\mathcal{M}_{k-1}$, which in our theory will be defined as the adjoint of $\mathcal{I}_k$, denoted by $\mathcal{I}_k^t$, with respect to the discrete L^2-product defined by (3.7) for nonquasiuniform triangulation or defined by (4.3) for quasiuniform triangulations, i.e. $\mathcal{I}_k^t : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$, satisfies

$$(\mathcal{I}_k^t u_k, v_{k-1})_{k-1} = (u_k, \mathcal{I}_k v_{k-1})_k, \quad \forall u_k \in \mathcal{M}_k, v_{k-1} \in \mathcal{M}_{k-1}.$$

Notice that the action of $\mathcal{I}_k^t$ results from inverting a diagonal matrix.

Another very important operator in our theory is the adjoint of $\mathcal{I}_k$, denoted by $\mathcal{I}_k^*$, with respect to the inner product $A(\cdot,\cdot)$, i.e. $\mathcal{I}_k^* : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ satisfying

$$A(\mathcal{I}_k^* u_k, v_{k-1}) = A(u_k, \mathcal{I}_k v_{k-1}), \quad \forall u_k \in \mathcal{M}_k, v_{k-1} \in \mathcal{M}_{k-1}.$$

The following obvious identity is useful in our later analysis:

$$A(\mathcal{I}_k^* u_k, v_k) = A(u_k, \mathcal{I}_k P_{k-1} v_k), \quad \forall u_k \in \mathcal{M}_k, v_k \in \mathcal{M}_k. \tag{4.1}$$

By the definition, the following relation holds:

$$\mathcal{I}_k^* = A_{k-1}^{-1} \mathcal{I}_k^t A_k.$$

In the case of nested meshes, $\mathcal{I}_k^*$ is just the Galerkin projection P_{k-1} if $\mathcal{I}_k$ is the natural inclusion. Hence we would expect $\mathcal{I}_k^*$ to have properties similar to those of P_{k-1}.

One of the main purposes of this section is to properly chose the prolongation operator $\mathcal{I}_k$ and to show the following so-called *regularity and approximation* property:

$$|A((I - \mathcal{I}_k \mathcal{I}_k^*)v, v)| \lesssim (\lambda_k^{-1} \|A_k v\|_k^2)^\beta A(v, v)^{1-\beta}, \quad \forall v \in \mathcal{M}_k, \tag{4.2}$$

where $\beta \in (0, 1]$ is a constant.

The regularity and approximation property of the form (4.2) will play a crucial role in our multigrid theory. As is meant by its name, it is closely related to the elliptic regularity of the underlying problem and the approximation property of the finite element spaces.

4.1 Nested Quasiuniform Meshes

In this section, we shall confine ourselves to domains that can be triangulated by a sequence of nested meshes. For Dirichlet problems, this essentially assumes that the domain is a polygon in $\mathbb{R}^2$ and tetrahedral in higher dimensions, since it seems impossible that a general domain with curved boundary can be partitioned so that a sequence of nested spaces can be constructed.

Therefore we assume that Ω has been triangulated with a nested sequence of quasi-uniform triangulations $\Omega = \cup_i \tau_k^i$ of size h_k for $k = 1, \ldots, j$ where the quasi-uniformity constants are independent of k (cf. [36]). These triangulations should be nested in the sense that any triangle τ_{k-1}^l can be written as a union of triangles of $\{\tau_k^i\}$. We further assume that there is a constant $\eta > 1$, independent of k, such that

$$h_k \asymp \eta^{-k}.$$

We recall the discrete L^2 inner product $(\cdot, \cdot)_k$ on $\mathcal{M}_k$ is defined by

$$(u, v)_k = h_k^d \sum_{x \in \mathcal{N}_k} u(x) v(x). \tag{4.3}$$

We shall take $\mathcal{I}_k$ to be the natural inclusion in this case, hence $\mathcal{I}_k^* = P_{k-1}$ as we mentioned above.

4.1.1 Error Estimate for the Galerkin Projection

Our first result is taken from Bramble and Pasciak [21], which verifies (4.2)

Theorem 4.1 *Assume α is as in* (3.5)*. Then* (4.2) *holds with $\beta = \alpha$. Namely*

$$A((I - P_{k-1})u, u) \lesssim (\lambda_k^{-1} \|A_k u\|_k^2)^\alpha A(u, u)^{1-\alpha}, \quad \forall u \in \mathcal{M}_k. \tag{4.4}$$

Proof. Let $u \in \mathcal{M}_k$, applying Schwarz's inequality and Lemma 3.7 gives

$$\begin{aligned} A((I - P_{k-1})u, u) &\leq (A_k^{\frac{1+\alpha}{2}} u, A_k^{\frac{1-\alpha}{2}} (I - P_{k-1})u)_k \\ &\leq \|A_k^{\frac{1+\alpha}{2}} u\|_k \|A_k^{\frac{1-\alpha}{2}} (I - P_{k-1})u\|_k \\ &\leq \|A_k^{\frac{1+\alpha}{2}} u\|_k \|(I - P_{k-1})u\|_{H^{1-\alpha}(\Omega)}. \end{aligned}$$

By Hölder's inequality,

$$\|A_k^{\frac{1+\alpha}{2}} u\|_k \leq \left(A(u, u)^{1-\alpha} \|A_k u\|_k^{2\alpha}\right)^{1/2}. \tag{4.5}$$

By Theorem 3.8, $\lambda_k \lesssim h_k^{-2}$ and therefore the theorem results from combining above inequalities with Proposition 3.8. ■

4.1.2 The Role of the Elliptic Regularity

It is well known that estimates of the form of (3.5) do not in general hold for $\alpha = 1$ and the range of α's for which they hold depends upon the regularity of the coefficients defining $\mathcal{L}$ and the smoothness of $\partial\Omega$. A frequently asked question is if the elliptic regularity is also crucial to the discrete approximation result like (4.4). The following result provides some answers to this question.

Theorem 4.2 *The following are equivalent*

1. (3.5) *holds for $\alpha = 1$ (full pick-up).*

2. $\|(I-P_{k-1})u\|_{H^1(\Omega)} \lesssim h_k\|A_k u\|_k, \quad \forall u \in \mathcal{M}_k.$

3. $\|(I-P_{k-1})u\|_k \lesssim h_k^2\|A_k u\|_k, \quad \forall u \in \mathcal{M}_k.$

4. $\|(I-P_{k-1})u\|_k \lesssim h_k\|A_k^{\frac{1}{2}} u\|_k, \quad \forall u \in \mathcal{M}_k.$

5. $\|P_{k-1}u\|_k \lesssim \|u\|_k, \quad \forall u \in \mathcal{M}_k.$

6. $\|A_k P_{k-1}u\|_k \lesssim \|A_k u\|_k, \quad \forall u \in \mathcal{M}_k.$

The most interesting and also most nontrivial part of this theorem is perhaps that (2) $\Rightarrow$ (1), which was first observed by Decker, Mandel and Parter [39]. The proof given below however was obtained by the author independently. The estimate 5) seems a little subtle; it is very tempting to assume 5) without assuming 1).

Proof. We will show that

$$1) \Leftrightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 5) \Rightarrow 6) \Rightarrow 2).$$

In Theorem 4.1, we have proven that (4.4) is a consequence of (3.5) for any α, hence 1) $\Rightarrow$ 2). we next show that 2) $\Rightarrow$ 1).

To continue the proof, we need to define $\mathcal{M}_k$ for all $k \geq 1$. For $k \leq j$, these spaces are defined in the beginning of this section. For $k > j$, we can define the spaces $\mathcal{M}_k$ in a similar fashion by induction. It is well-known that, on $H_0^1(\Omega)$, for any fixed m

$$P_m + \sum_{k=m+1}^{\infty}(P_k - P_{k-1}) = \lim_{k\to\infty} P_k = I,$$

hence, if u is the solution of (3.1)(for any given $f \in L^2(\Omega)$),

$$\begin{aligned}(I-P_m)u &= \sum_{k=m+1}^{\infty}(P_k - P_{k-1})u \\ &= \sum_{k=m+1}^{\infty}(I-P_{k-1})P_k u,\end{aligned}$$

where we have used a fact that $P_{k-1}P_k = P_{k-1}$.

Taking the H^1-norm on both hand sides of above identity and applying triangle inequality and assertion (2), we get

$$\begin{aligned}\|(I-P_m)u\|_{H^1(\Omega)} &\le \sum_{k=m+1}^{\infty} \|(I-P_{k-1})P_k u\|_{H^1(\Omega)} \\ &\lesssim \sum_{k=m+1}^{\infty} h_k \|A_k P_k u\|_k.\end{aligned}$$

By definition, $\forall v \in \mathcal{M}_k$,

$$(A_k P_k u, v)_k = A(P_k u, v) = (f, v) \lesssim \|f\|_{L^2(\Omega)} \|v\|_k,$$

which implies that

$$\|A_k P_k u\|_k \lesssim \|f\|_{L^2(\Omega)}.$$

Consequently

$$\|(I-P_m)u\|_{H^1(\Omega)} \lesssim \sum_{k=m+1}^{\infty} h_k \|A_k P_k u\|_k$$

$$\lesssim \sum_{k=m+1}^{\infty} h_k \|f\|_{L^2(\Omega)} \lesssim h_m \|f\|_{L^2(\Omega)},$$

since

$$\sum_{k=m+1}^{\infty} h_k \lesssim h_m \sum_{k=1}^{\infty} \eta^{-k} \le h_m.$$

Therefore, we conclude that for any $m > 1, f \in L^2(\Omega)$ and the corresponding solution u of (3.1),

$$\|(I-P_m)u\|_{H^1(\Omega)} \lesssim h_m \|f\|_{L^2(\Omega)}. \tag{4.6}$$

It is known that this holds if and only if (3.5) holds for $\alpha = 1$, cf. Wahlbin [84]. This proves that (2) $\Rightarrow$ (1).

To see 2) $\Rightarrow$ 3), we use a duality argument. Let $v = (I - P_{k-1})u$,

$$
\begin{aligned}
\|(I - P_{k-1})u\|_k^2 &= ((I - P_{k-1})u, v)_k \\
&\le A((I - P_{k-1})u, A_k^{-1}v) \le A((I - P_{k-1})u, (I - P_{k-1})A_k^{-1}v) \\
&\le \|(I - P_{k-1})u\|_{H^1(\Omega)} \|(I - P_{k-1})A_k^{-1}v\|_{H^1(\Omega)} \\
&\lesssim h_k^2 \|A_k u\|_k \|v\|_k,
\end{aligned}
$$

which implies 3).

It is very straightforward to verify that 3) $\Rightarrow$ 4) $\Rightarrow$ 5), since, by Theorem (3.8),

$$
\|A_k^{\frac{1}{2}} u\|_k \lesssim h_k^{-1} \|u\|_k, \quad \forall u \in \mathcal{M}_k.
$$

To show 5) $\Rightarrow$ 6), setting $v = A_k P_{k-1} u$, we have

$$
\begin{aligned}
\|A_k P_{k-1} u\|_k^2 &= (A_k P_{k-1} u, v)_k = (A_k u, P_{k-1} v)_k \\
&\le \|A_k u\|_k \|P_{k-1} v\|_k \lesssim \|A_k u\|_k \|v\|_k
\end{aligned}
$$

which implies 6).

It remains to show that 6) $\Rightarrow$ 2). To do this, using the L^2 projection Q_{k-1} defined in Section 3.8 and applying Theorem 3.5, we get

$$
\begin{aligned}
\|(I - P_{k-1})u\|_{H^1(\Omega)}^2 &\lesssim A((I - P_{k-1})u, (I - P_{k-1})u) \\
&= A((I - Q_{k-1})(I - P_{k-1})u, (I - P_{k-1})u) \\
&\lesssim \|(I - Q_{k-1})(I - P_{k-1})u\|_{L^2(\Omega)} \|A_k (I - P_{k-1})u\|_k \\
&\lesssim h_k \|(I - P_{k-1})u\|_{H^1(\Omega)} \|A_k u\|_k,
\end{aligned}
$$

as desired. This completes the proof. ∎

4.1.3 More Estimates for the Weighted L^2 Projection

The weighted L^2 projection was defined and analyzed in Section 3.9 of the previous chapter, where we have obtained some results for weighted L^2 projection under an additional constraint that there is no crossing point on the interfaces. This constraint will be removed here in some special circumstances.

Our main result is devoted to two dimensional problems.

Theorem 4.3 *Assume $d = 2$ and $\{\mathcal{T}_k, k \in \mathcal{I}\}$ is quasiuniform. Then for any $\tau \in \mathcal{T}_k$*

$$\|(I - I_k)u\|_{L^2(\tau)} \lesssim h_k(\log \frac{h_k}{h_j})^{\frac{1}{2}} |u|_{H^1(\tau)} \quad \forall u \in \mathcal{M}_j.$$

where $I_k : \mathcal{M}_j \mapsto \mathcal{M}_k$ is the interpolation operator which was defined in Section 3.6 in Chapter 3. Consequently,

$$\|(I - Q_k^{\omega})u\|_{L^2_{\omega}(\Omega)} \leq \|(I - I_k)u\|_{L^2_{\omega}(\Omega)} \lesssim h_k(\log \frac{h_k}{h_j})^{\frac{1}{2}} |u|_{H^1_{\omega}(\Omega)} \quad \forall u \in \mathcal{M}_j.$$

Proof. Let $\hat{\tau}$ be the standard reference element, then

$$\|(I - I_k)u\|_{L^2(\tau)} \lesssim h_k \|(\hat{I} - \hat{I}_k)\hat{u}\|_{L^2(\hat{\tau})}.$$

It follows from the discrete Sobolev inequality (3.24) that

$$\begin{aligned} \|(\hat{I} - \hat{I}_k)\hat{u}\|_{L^2(\hat{\tau})} &\leq \|\hat{u}\|_{L^\infty(\hat{\tau})} \\ &\lesssim (\log \frac{h_k}{h_j})^{\frac{1}{2}} \|\hat{u}\|_{H^1(\hat{\tau})}. \end{aligned}$$

Replacing $\hat{u}$ by $\hat{u} + c$ for any constant c, we have

$$\|(\hat{I} - \hat{I}_k)\hat{u}\|_{L^2(\hat{\tau})} \lesssim (\log \frac{h_k}{h_j})^{\frac{1}{2}} \inf_{c \in \mathbb{R}^1} \|\hat{u} + c\|_{H^1(\hat{\tau})}$$

$$\lesssim (\log \frac{h_k}{h_j})^{\frac{1}{2}} |\hat{u}|_{H^1(\hat{\tau})} \lesssim (\log \frac{h_k}{h_j})^{\frac{1}{2}} |u|_{H^1(\tau)}.$$

The desired result then follows. ■

If the space we are considering is somehow "close" to $\mathcal{M}_k$, stronger results can then be obtained. For example, we can use the same argument to show the following

Lemma 4.1 *The following holds in any number of dimension:*

$$\|(I - Q_k^{\omega})u\|_{L^2_\omega(\Omega)} \lesssim \|(I - I_k)u\|_{L^2_\omega(\Omega)} \lesssim h_k|u|_{H^1_\omega(\Omega)}, \quad \forall u \in \mathcal{M}_{k+1}.$$

4.2 Nested Nonquasiuniform Meshes

Assuming we are given a nested sequence of triangulations $\{\mathcal{T}_k, k \in \mathcal{I}\}$, which are not necessarily quasiuniform but still satisfy the basic assumption (**A2.1**). As usual we have the corresponding finite element spaces $\{\mathcal{M}_k, k = 1, \ldots, j\}$.

For any $x \in \Omega$, we define a local mesh size

$$h_{k,x} = \frac{1}{|A_{k,x}|} \sum_{\tau \in A_{k,x}} h_\tau$$

where $A_{k,x} = \{\tau \in \mathcal{T}_k : x \in \tau\}$ and $|A_{k,x}|$ is the number of elements in $A_{k,x}$.

We need to assume that any consecutive meshes are comparatively close in the sense that

$$h_{k-1,x} \lesssim h_{k,x} \le h_{k-1,x}, \quad \forall k, x.$$

Roughly speaking, the number of elements of $\mathcal{T}_k$ contained in any element of $\mathcal{T}_{k-1}$ is bounded.

The discrete inner product $(\cdot, \cdot)_k$ will be still defined as in (3.7). It is trivial to see that the induced norm $\|\cdot\|_k$ satisfies:

$$\|u\|_k{}^2 \asymp h_k^2 \sum_{\tau \in \mathcal{T}_k} h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} |u(x)|^2, \tag{4.7}$$

where $h_\tau = diam(\tau)$.

Lemma 4.2 *For any $v \in \mathcal{M}_k$,*

$$\|(I - I_{k-1})v\|_k{}^2 \lesssim h_k^2 A(v, v).$$

Proof. For any $\tau \in \mathcal{T}_{k-1}$, it is routine to show that

$$\|(I - I_{k-1})v\|^2_{L^2(\tau)} \lesssim h_\tau^2 |v|^2_{H^1(\tau)}, \quad \forall v \in \mathcal{M}_k,$$

where $h_\tau = diam(\tau)$.

But

$$h_\tau^d \sum_{x \in \mathcal{N}_k \cap \tau} |(I - I_{k-1})v(x)|^2 \lesssim \|(I - I_{k-1})v\|^2_{L^2(\tau)},$$

and hence

$$h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} |(I - I_{k-1})v(x)|^2 \lesssim |v|^2_{H^1(\tau)}.$$

Summing over all $\tau \in \mathcal{T}_{k-1}$, we then get

$$\sum_{\tau \in \mathcal{T}_{k-1}} h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} |(I - I_{k-1})v(x)|^2 \lesssim A(v, v).$$

The desired result then follows because of (4.7). ■

As we see that the factor h_k^2 in the definition of $(\cdot, \cdot)_k$ plays no role in the above proof, it appears there because of the way we define $(\cdot, \cdot)_k$.

4.3 Specially Coupled Nonnested Grids on Curved Boundary Domains

In general, it is not hard to see that a domain with a curved boundary can not be partitioned by a sequence of nested grids which successively more closely approximate the domain. Hence we have to use nonnested meshes. The way to set up the grids is of course not unique. The triangulations we will discuss first are almost nested ones, namely the nonnestedness only occurs near the boundary.

For simplicity of exposition, we assume the domain Ω is two dimensional and convex. The techniques to be presented extend to higher dimensions in a straightforward way. A little bit more work seems to be needed for the nonconvex case; we did not carry out the details however.

We first define the sequence of grids and their corresponding subspaces. We start with a coarse grid triangulation $\mathcal{T}_1$. If $\mathcal{T}_{k-1}$ is defined, we then define the finer grid $\mathcal{T}_k$ by refining each triangle $\tau \in \mathcal{T}_{k-1}$ according to the following rules:

1. If there are two vertices of τ lying on $\partial\Omega$, one of its edges would cut off an arc from $\partial\Omega$. τ is then divided into four triangles by connecting the midpoint of the boundary arc and the midpoints of the other two edges of τ.

2. Otherwise τ is divided into four triangles as usual by connecting the midpoints of all its edges.

As soon as $\mathcal{T}_k$'s have been constructed, the $\mathcal{M}_k$'s are then defined as usual to be the set of functions which are piecewise linear on $\mathcal{T}_k$ and vanish at the nodes of $\mathcal{T}_k$ on $\partial\Omega$.

We make the following two hypotheses which depend on the choice of $\mathcal{T}_1$ and the shape of $\partial\Omega$:

(H.1) All $\mathcal{T}_k$'s are quasiuniform of size h_k satisfying

$$h_k \asymp \eta^{-k},$$

for some constant $\eta \geq 2$ and the constants of quasiuniformity are independent of k.

(H.2) dist$\{\Omega_k, \partial\Omega\} \lesssim h_k^2$ where

$$\Omega_k = \cup_{\tau \in \mathcal{T}_k} \tau.$$

In fact, η should be 2 in $(H.1)$ by our construction. $(H.2)$ is a weak constraint on the multilevel triangulations. We will see that the most crucial point here is the way the triangulations are obtained.

Even though the $\mathcal{M}_k$'s are not nested, they can be thought of as subspaces of $H_0^1(\Omega)$ by extending the functions in $\mathcal{M}_k$ to be zero in $\Omega \setminus \Omega_k$. Hence $A(\cdot,\cdot)$ is well defined on all $\mathcal{M}_k$'s and prolongation operators $\mathcal{I}_k : \mathcal{M}_{k-1} \mapsto \mathcal{M}_k$ are taken to be the standard nodal value interpolants, namely

$$(\mathcal{I}_k v)(x) = v(x), \quad \forall x \in \mathcal{N}_k. \tag{4.8}$$

The discrete product on $\mathcal{M}_k$ is defined by

$$(u, v)_k = h_k^d \sum_{x \in \mathcal{N}_k} (uv)(x).$$

The key role playing in our analysis is the following super-approximation property:

Lemma 4.3

$$\|(I - \mathcal{I}_k)v\|_{H^1} \lesssim h_k^{1/2} \|v\|_{H^1}, \quad \forall v \in \mathcal{M}_{k-1}.$$

Proof. To illustrate the main idea, we present the proof for the two dimensional case. By the construction of the triangulation, we only need to give the estimate on the elements near the boundary since $v - I_k v$ vanishes otherwise.

Take a boundary element $\Delta_{123} \in \mathcal{T}_{k-1}$ as shown in the above picture from which four elements in $\mathcal{T}_k$ are obtained by the described refinement process (note that a_7 is note a node). Let D denote the set consisting of these four elements and let

$a_i = (\xi_i, \eta_i)$, $i = 1, \cdots, 7$ be as shown in the picture. Obviously, it sufficient to establish the following estimate:

$$|v - I_k v|^2_{H^1(D)} \lesssim h_k |v|^2_{H^1(\Delta_{123})}. \tag{4.9}$$

An elementary calculation shows that, on Δ_{123}

$$\frac{\partial}{\partial x_1} v(x) = \frac{\eta_1 - \eta_2}{A_{123}} v(a_3),$$

and on Δ_{246}

$$\frac{\partial}{\partial x_1}(I_k v)(x) = \frac{\eta_7 - \eta_2}{2A_{247}} v(a_3).$$

where A_{123} and A_{247} are the areas of Δ_{123} and Δ_{247} respectively.

Notice that

$$A_{123} = 4A_{247}(1 + O(h_k)),$$

$$\eta_1 - \eta_2 = 2(\eta_7 - \eta_2)(1 + O(h_k)).$$

Thus on Δ_{247}

$$\begin{aligned} \frac{\partial}{\partial x_1}(v - I_k v)(x) &= \frac{\eta_1 - \eta_2}{A_{123}} v(a_3) - \frac{\eta_7 - \eta_2}{2A_{247}} v(a_3) \\ &= \frac{\eta_1 - \eta_2}{A_{123}} O(h_k) v(a_3) = O(1) v(a_3) \end{aligned}$$

and on Δ_{276}

$$\begin{aligned} \frac{\partial}{\partial x_1}(v - I_k v)(x) &= -\frac{\partial}{\partial x_1}(I_k v)(x) = -\frac{\eta_7 - \eta_2}{2A_{247}} O(h_k) v(a_3) \\ &= O(h_k^{-1}) v(a_3). \end{aligned}$$

Therefore

$$\int_{\Delta_{246}} |\frac{\partial}{\partial x_1}(v - I_k v)(x)|^2 dx = (\int_{\Delta_{247}} + \int_{\Delta_{276}}) |\frac{\partial}{\partial x_1}(v - I_k v)(x)|^2 dx$$

$$\lesssim \quad (A_{247} + A_{276} h_k^{-2}) v^2(a_3)$$

$$\lesssim \quad h_k v^2(a_3) \lesssim h_k |v|^2_{H^1(\Delta_{123})}.$$

The proof for $\frac{\partial}{\partial x_2}$ is analogous, hence

$$|v - I_k v|^2_{H^1(\Delta_{246})} \lesssim h_k |v|^2_{H^1(\Delta_{123})}.$$

Similarly

$$|v - I_k v|^2_{H^1(\Delta_{165})} \lesssim h_k |v|^2_{H^1(\Delta_{123})}.$$

For $|v - I_k v|_{H^1(\Delta_{456})}$, we can easily use the same technique to derive the following better estimate

$$|v - I_k v|^2_{H^1(\Delta_{456})} \lesssim h_k^2 |v|^2_{H^1(\Delta_{123})}.$$

There is nothing to estimate on Δ_{354} since $v - I_k v$ is zero there. The estimate 4.9 then follows.

This completes the proof. ■

As a direct consequence of Lemma (4.3), by using triangle inequality, we have

Lemma 4.4 *There exists a constant $C > 0$, independent of k such that*

$$A(\mathcal{I}_k v, \mathcal{I}_k v) \le (1 + C h_k^{1/2}) A(v, v), \quad \forall v \in \mathcal{M}_{k-1}.$$

Consequently,

$$A(\mathcal{I}_k^* v, \mathcal{I}_k^* v) \le (1 + C h_k^{1/2}) A(v, v), \quad \forall v \in \mathcal{M}_k.$$

Lemma 4.5 (4.2) *holds for $\beta = \min(\frac{\alpha}{2}, \frac{1}{4})$*

Proof. An elementary manipulation (using (4.1)) yields

$$\begin{aligned} & A((I - \mathcal{I}_k \mathcal{I}_k^*) v, v) \\ & \quad = A((I - \mathcal{I}_k) \mathcal{I}_k^* v, v) + A(v, (I - \mathcal{I}_k P_{k-1}) v) \\ & \quad = A((I - \mathcal{I}_k) \mathcal{I}_k^* v, v) + A(v, (I - \mathcal{I}_k) P_{k-1} v) + A(v, (I - P_{k-1}) v) \\ & \quad = A((I - \mathcal{I}_k)(\mathcal{I}_k^* + P_{k-1}) v, v) + A(v, (I - P_{k-1}) v). \end{aligned}$$

It follows from Lemmas 4.3 and 4.4 that

$$\begin{aligned}&|A((I-\mathcal{I}_k)(\mathcal{I}_k^*+P_{k-1})v,v)|\\ &\qquad \lesssim h_k^{\frac{1}{2}}A(v,v)\lesssim(\lambda_k^{-1}\|A_kv\|_k{}^2)^{\frac{1}{4}}A(v,v)^{\frac{3}{4}}.\end{aligned}$$

Using Proposition 3.8 and triangle inequality, we can show that

$$\|P_k(I-P_{k-1})v\|_{H^{1-\alpha}(\Omega)}\lesssim h_k^{\alpha}\|v\|_{H^1(\Omega)}.$$

Consequently, from (3.52) and (4.5), we deduce that

$$\begin{aligned}A(v,(I-P_{k-1})v)&=A(v,P_k(I-P_{k-1})v)\\ &=(A_k^{\frac{1-\alpha}{2}}P_k(I-P_{k-1})v,A_k^{\frac{1+\alpha}{2}}v)_k\\ &\lesssim\|P_k(I-P_{k-1})v\|_{H^{1-\alpha}(\Omega)}\|A_k^{\frac{1+\alpha}{2}}v\|_k\\ &\lesssim(\lambda_k^{-1}\|A_kv\|_k{}^2)^{\frac{\alpha}{2}}A(v,v)^{1-\frac{\alpha}{2}}.\end{aligned}$$

The desired result then follows. ■

4.4 Loosely Coupled Nonnested Grids on Curved Boundary Domains

The nonnested described in the previous section are not very flexible in applications. We will next study some more general triangulations.

> *In this section, the multilevel triangulation $\{\mathcal{T}_k, k=1,\cdots,j\}$ are assumed to satisfy both (H.1) and (H.2) from the previous section.*

The point is that we need not specify the way in which the triangulations are obtained, or how the different spaces should be related. Hypothesis (H.1) and (H.2) seem to be the minimal requirements in the theory of nonnested multigrid methods on quasiuniform grids, since, on the one hand, the spaces need certain approximation property to guarantee the convergence of the algorithm, and on the other hand, they need to possess certain asymptotic properties so that the complexity of the algorithm is desirable.

4.4.1 Using the interpolant as a prolongation: $d = 2$

As we mentioned before, the interpolation operator is a very natural choice for *prolongation*; we will provide estimates for this choice in plane domain. The arguments given below do not seem to apply in higher dimensions. In that case we will use a different prolongation which will be discussed in the next subsection.

Lemma 4.6 (4.2) *holds for* $\beta = \min(\frac{\alpha}{2}, \frac{1}{4})$

Proof. Without loss of generality, we assume that $\alpha < \frac{1}{2}$. It follows from Lemma (3.7) that

$$A((I - \mathcal{I}_k\mathcal{I}_k^*)v, v) \leq \|A_k^{\frac{1+\beta}{2}} v\|_k \|(I - \mathcal{I}_k\mathcal{I}_k^*)v\|_{H^{1-\beta}(\Omega)}.$$

Let us first quote the result in Theorem (3.3) that

$$\|(I - \mathcal{I}_k)v\|_{H^{1-\beta}(\Omega)} \lesssim h_k^{\beta+\delta}\|v\|_{H^{1+\delta}(\Omega)}, \tag{4.10}$$

for $0 \leq \beta \leq 1$ and $0 < \delta \leq 1$. By the fractional inverse inequality, (4.10) also holds for $\delta = 0$ when $v \in \mathcal{M}_{k-1}$. Consequently

$$\|\mathcal{I}_k v_{k-1}\|_{H^1(\Omega)} \lesssim \|v_{k-1}\|_{H^1(\Omega)}, \quad \forall v_{k-1} \in \mathcal{M}_{k-1}$$

which can be rewritten as

$$A(\mathcal{I}_k v_{k-1}, \mathcal{I}_k v_{k-1}) \lesssim A(v_{k-1}, v_{k-1}), \quad \forall v_{k-1} \in \mathcal{M}_{k-1}.$$

Thus

$$A(\mathcal{I}_k^* v_k, \mathcal{I}_k^* v_k) \lesssim A(v_k, v_k), \quad \forall v_k \in \mathcal{M}_k. \tag{4.11}$$

The triangle inequality gives that

$$\|(I-\mathcal{I}_k\mathcal{I}_k^*)v\|_{H^{1-\beta}(\Omega)} \le \|(I-P_{k-1})v\|_{H^{1-\beta}(\Omega)} + \|(I-\mathcal{I}_k)\mathcal{I}_k^* v\|_{H^{1-\beta}(\Omega)} + \|(\mathcal{I}_k^*-P_{k-1})v\|_{H^{1-\beta}(\Omega)}. \quad (4.12)$$

The estimate of the first term of (4.12) is provided by Proposition 3.8, namely

$$\|(I-P_{k-1})v\|_{H^{1-\beta}(\Omega)} \lesssim h_k^{\beta}\|v\|_{H^1(\Omega)}.$$

To estimate the second term of (4.12), we deduce from (4.10) and (4.11) that

$$\|(I-\mathcal{I}_k)\mathcal{I}_k^* v\|_{H^{1-\beta}(\Omega)} \lesssim h_k^{\beta}\|\mathcal{I}_k^* v\|_{H^1(\Omega)} \lesssim h_k^{\beta}\|v\|_{H^1(\Omega)}.$$

It remains to estimate the last term of (4.12); we use a duality argument. For $\phi \in C_0^\infty(\Omega)$, let $w \in H_0^1(\Omega)$ satisfy

$$\mathcal{L}w = \phi,$$

then, it follows from (4.10), Proposition 3.9 and (3.5) that

$$\begin{aligned} ((\mathcal{I}_k^*-P_{k-1})v,\phi) &= A((\mathcal{I}_k^*-P_{k-1})v,w) = A(v,(\mathcal{I}_k-I)P_{k-1}w) \\ &\le \|v\|_{H^1(\Omega)}\|(\mathcal{I}_k-I)P_{k-1}w\|_{H^1(\Omega)} \lesssim h_k^{\beta}\|P_{k-1}w\|_{H^{1+\beta}(\Omega)}\|v\|_{H^1(\Omega)} \\ &\lesssim h_k^{\beta}\|w\|_{H^{1+\beta}(\Omega)}\|v\|_{H^1(\Omega)} \lesssim h_k^{\beta}\|\phi\|_{H^{\beta-1}(\Omega)}\|v\|_{H^1(\Omega)}. \end{aligned}$$

Consequently

$$\begin{aligned} \|(\mathcal{I}_k^*-P_{k-1})v\|_{H^{1-\beta}(\Omega)} &\lesssim \sup_{\phi\in C_0^\infty(\Omega)} \frac{((\mathcal{I}_k^*-P_{k-1})v,\phi)}{\|\phi\|_{H^{\beta-1}(\Omega)}} \\ &\lesssim h_k^{\beta}\|v\|_{H^1(\Omega)}. \end{aligned}$$

The proof is then complete. ■

4.4.2 Using the L^2 quasi-projection as a prolongation: $d \geq 2$

As we mentioned above we have difficulties in the analysis if we use the interpolation as prolongation for higher dimensional problem since in this case (4.10) fails to hold. In the following, we will use a new type prolongation operator which theoretically works fine in any number of dimensions.

We will choose the discrete inner product $(\cdot, \cdot)_k$ on $\mathcal{M}_k$ in such a way that

$$|(u,v)_k - (u,v)| \lesssim h_k \|u\|_{H^1(\Omega)} \|v\|_k, \quad \forall u, v \in \mathcal{M}_k.$$

We will choose prolongation operator $\mathcal{I}_k$ to be L^2 quasi-projection Π_k as defined in Section 3.10. In this case, the restriction turns out to be Π_{k-1} since by definition, we have

$$(\Pi_k u, v)_k = (u, \Pi_{k-1} v)_{k-1} \quad \forall u \in \mathcal{M}_k, v \in \mathcal{M}_{k-1}.$$

Lemma 4.7 *For any $u \in \mathcal{M}_k$, we have*

$$\begin{aligned} \|(I - \mathcal{I}_k^*)u\|_{H^1(\Omega)} &\lesssim h_k^\alpha \|A_k^{\frac{1+\alpha}{2}} u\|_k \\ &\lesssim (h_k \|A_k u\|_k)^\alpha \|u\|_{H^1(\Omega)}^{1-\alpha}. \end{aligned}$$

Consequently,

$$\|\mathcal{I}_k^* u\|_{H^1(\Omega)} \lesssim \|u\|_{H^1(\Omega)}.$$

Proof. Using 4.1, we have

$$\begin{aligned} A((I - \mathcal{I}_k^*)u, v) &= A(u, (I - \mathcal{I}_k P_{k-1})v) \\ &= (A_k u, (P_k - \mathcal{I}_k P_{k-1})v)_k \\ &= \|A_k^{\frac{1+\alpha}{2}} u\|_k \|A_k^{\frac{1-\alpha}{2}} (P_k - \mathcal{I}_k P_{k-1})v\|_k. \end{aligned}$$

Applying Lemmas 3.7 and 3.6,

$$\|A_k^{\frac{1-\alpha}{2}} (P_k - \mathcal{I}_k P_{k-1})v\|_k \leq \|(P_k - \mathcal{I}_k P_{k-1})v\|_{H^{1-\alpha}(\Omega)}$$

$$\lesssim \|(I-P_k)v\|_{H^{1-\alpha}(\Omega)} + \|\mathcal{I}_k(I-P_{k-1})v\|_{H^{1-\alpha}(\Omega)} + \|(I-\mathcal{I}_k)v\|_{H^{1-\alpha}(\Omega)}$$

$$\lesssim h_k^\alpha \|v\|_{H^1(\Omega)}.$$

Combining the above two inequalities with $v = (I - \mathcal{I}_k^*)u$ yields the desired result. ■

Lemma 4.8 *For any $u \in \mathcal{M}_k$, we have*

$$A((I - \mathcal{I}_k\mathcal{I}_k^*)u, u) \lesssim \left(\lambda_k^{-1}\|A_k u\|_k\right)^{\frac{\alpha}{2}} A(u,u)^{1-\frac{\alpha}{2}}.$$

Proof. By Lemma 3.6

$$\begin{aligned} A((I-\mathcal{I}_k)u,u) \le (A_k u, (I-\mathcal{I}_k)u)_k &\lesssim h_k \|A_k u\|_k \|u\|_{H^1(\Omega)} \\ &\lesssim \left(\lambda_k^{-1}\|A_k u\|_k\right)^{\frac{\alpha}{2}} A(u,u)^{1-\frac{\alpha}{2}} \end{aligned}$$

and it follows from 3.44 in Lemma 3.6 and Lemma 4.7 that

$$\begin{aligned} A(\mathcal{I}_k(I-\mathcal{I}_k^*)u,u) &\lesssim \|(I-\mathcal{I}_k^*)u\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)} \\ &\lesssim \left(\lambda_k^{-1}\|A_k u\|_k\right)^{\frac{\alpha}{2}} A(u,u)^{1-\frac{\alpha}{2}}. \end{aligned}$$

The desired result is obtained by adding up the above two estimates. ■

4.5 An Estimate for a Nonconforming Element

Given a quasiuniform triangulation $\{\mathcal{T}_k, k = 1, \cdots, j\}$, as usual, let $\mathcal{M}_k$ be a subspace of of $H_0^1(\Omega)$ consisting of piecewise continuous functions on $\mathcal{T}_k$. And on the same triangulation $\mathcal{T}_k$, we will define a finite element space $\bar{\mathcal{M}}_k$ which is a space of piecewise linear functions on $\mathcal{T}_k$ that assume the same value at two adjacent elements at the midpoint of their common edge and vanish at the midpoint of the edge on the boundary of Ω. Since $\bar{\mathcal{M}}_k \not\subset H_0^1(\Omega)$, it is usually called nonconforming. This element was first proposed by Crouzeix and Raviart [37] for second order elliptic equation, it can also be applied to the Stokes equation. The estimate given in this section will be used to analyze a multigrid algorithm to solve the described nonconforming finite element equation.

The main idea is to use $\mathcal{M}_k$ as a "coarse space" of $\bar{\mathcal{M}}_k$ because of the following trivial but important fact:

$$\mathcal{M}_k \subset \bar{\mathcal{M}}_k.$$

Since the usual bilinear form $A(\cdot,\cdot)$ is not defined on $\bar{\mathcal{M}}_k$, we define

$$A_k(u,v) = \sum_{\tau\in\mathcal{T}_k} A_\tau(u,v) \quad \forall u,v \in \bar{\mathcal{M}}_k.$$

Define $\pi_k : \bar{\mathcal{M}}_k \mapsto \mathcal{M}_k$ by

$$(\pi_k u)(x) = \begin{cases} 0, & \text{if } x \in \partial\Omega \cap \mathcal{N}_k; \\ \frac{1}{|\bar{\mathcal{N}}_x|}\sum_{y\in\bar{\mathcal{N}}_x} u(y), & \text{if } x \in \mathcal{N}_k \setminus \partial\Omega, \end{cases}$$

where $\bar{\mathcal{N}}_x$ is the set of midpoints of the edges with x as one of its endpoints.

The discrete L^2-inner product on $\bar{\mathcal{M}}_k$ is

$$(u,v)_k = h_k^2 \sum_{x\in\bar{\mathcal{N}}_k} u(x)v(x),$$

where $\bar{\mathcal{N}}_k$ = set of all midpoints of the edges in $\mathcal{T}_k$. and the discrete L^2-inner product on $\mathcal{M}_k$ is defined by

$$(u,v)_k = h_k^2 \sum_{x\in\mathcal{N}_k} u(x)v(x).$$

The main result of this section is as follows:

Lemma 4.9

$$\|(I-\pi_k)u\|^2 \lesssim h_k^2 A_k(u,u), \quad \forall u \in \bar{\mathcal{M}}_k.$$

Proof. For a given $x \in \bar{\mathcal{N}}_k \setminus \partial\Omega$, in the following, x_1 and x_2 will denote the endpoints of the edge where x is located. We have

$$\begin{aligned}
\|(I-\pi_k)u\|^2 &\lesssim h_k^2 \sum_{x\in\bar{\mathcal{N}}_k} |(I-\pi_k)u(x)|^2 \\
&= h_k^2 \sum_{x\in\bar{\mathcal{N}}_k} |u(x) - \frac{1}{2}[(\pi_k u)(x_1) + (\pi_k u)(x_2)]|^2 \\
&\lesssim h_k^2 \sum_{x\in\bar{\mathcal{N}}_k} \sum_{i=1}^{2} |u(x) - (\pi_k u)(x_i)|^2 \\
&\lesssim h_k^2 \sum_{x\in\bar{\mathcal{N}}_k} \sum_{i=1}^{2} \sum_{y\in\bar{\mathcal{N}}_{x_i}} |u(x) - u(y)|^2.
\end{aligned}$$

It is straightforward to check that

$$\sum_{y\in\bar{\mathcal{N}}_{x_i}} |u(x)-u(y)|^2 \leq \sum_{\tau\in\mathcal{T}_k, x\in\tau} \sum_{x',x''\in\bar{\mathcal{N}}_k\cap\tau} |u(x')-u(x'')|^2.$$

Consequently

$$\begin{aligned}
\|(I-\pi_k)u\|^2 &\lesssim h_k^2 \sum_{\tau\in\mathcal{T}_k} \sum_{x',x''\in\bar{\mathcal{N}}_k\cap\tau} |u(x')-u(x'')|^2 \\
&\lesssim h_k^2 A(u,u).
\end{aligned}$$

This completes the proof. ■

Chapter 5

Iterative Methods and Preconditioning

This chapter is devoted to a summary of a few facts on some linear iterative methods, preconditioners and the conjugate gradient method. Some linear iterative methods such as Richardson's and Gauss–Seidel or SOR method are often used as smoothers in the multigrid algorithms, hence they are of fundamental importance in the theory of multigrid methods. Preconditioned conjugate gradient methods will play a dominant role in the whole theory of this work, hence a rather detailed discussion is given for properties of preconditioners and the analysis of conjugate gradient methods.

5.1 Linear Iterative Methods

Let A be a SPD operator on a finite dimensional space $\mathcal{M}$ with respect to an inner product $(\cdot, \cdot)$. We are interested in solving the following equation on $\mathcal{M}$:

$$Au = b. \tag{5.1}$$

To obtain a linear iterative method for solving (5.1), we take a nonsingular operator S and rewrite (5.1) in the following equivalent form:

$$u = Ku + Sb,$$

where $K = I - SA$.

A linear iterative scheme can then be given by

$$u^\ell = Ku^{\ell-1} + Sb, \quad \ell = 1, 2, \cdots. \tag{5.2}$$

It is well-known that a necessary and sufficient condition that (5.2) converges for all initial guess u^0 is that

$$\rho(K) < 1, \tag{5.3}$$

where $\rho(K)$ is the spectral radius of K.

An obvious sufficient condition for (5.3) to be valid is that

$$\|K\| \le \delta < 1,$$

and moreover

$$\|u - u^l\| \le \delta^l \|u - u^0\|.$$

Another less obvious but very important sufficient condition for (5.3) to be valid is that is that $I - K^*K$ is a positive operator, where $*$ is the operator adjoint with respect to the inner product $(A\cdot, \cdot)$. Notice that $I - K^*K$ is self-adjoint with respect to $(A\cdot, \cdot)$. But we will be interested in the following stronger condition:

$$\|Au\|^2 \lesssim \rho(A)A((I - K^*K)u, u) \quad \forall u \in \mathcal{M}. \tag{5.4}$$

The above condition may not look very natural; it is a technical assumption required for the iterative method in order that it can be used as a smoother in a multigrid process. We will verify it for the Richardson method (which is trivial) and for the Gauss-Seidel method in the context of finite element equations.

Another variation of the Richardson iteration is the so–called *damped* Richardson method in which $S = \omega\lambda_{\max}^{-1} I$ for $\omega \in (0, 2)$. It is straightforward to verify that (5.4) holds for this method.

5.1.1 Richardson's Method

This method may be the simplest of all which is defined by (5.2) by choosing

$$S = \lambda_{max}^{-1} I,$$

where λ_{max} = the maximum eigenvalue of A. It is easy to see, if $A = A_k$ is the finite element operator described in Chapter 2, then

$$\|K\| = \|I - \lambda_{max}^{-1} A\| = 1 - \frac{\lambda_{min}}{\lambda_{max}} = 1 - ch_k^2 \equiv \delta < 1.$$

Therefore (5.3) is satisfied. The verification of (5.4) is trivial.

We see that δ here is very close to 1 if h_k is small. Hence this algorithm converges very slowly. However this is not the whole story of the algorithm; it has some very interesting properties which we will discuss now.

Assume $0 < \lambda_{\min} = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda_{\max}$ are the eigenvalues of A and $\{\phi_1, \phi_2, \cdots, \phi_n\}$ are the corresponding orthonormal eigenfunctions, then

$$(u - u^l, \phi_k) = (1 - \frac{\lambda_k}{\lambda_n})^l (u - u^0, \phi_k).$$

For $k = n$,

$$(u - u^l, \phi_n) = 0;$$

for $k = n - 1$,

$$(u - u^l, \phi_{n-1}) = (1 - \frac{\lambda_{n-1}}{\lambda_n})^l (u - u^0, \phi_{n-1})$$

which goes to zero rapidly as l approaches infinity.

We observed that this simple iteration, although it converges very slowly globally, reduces the error very rapidly in the directions corresponding to the top eigenvalues, namely the high frequencies are damped quickly. This simple observation is the key motivation upon which the multigrid has been developed.

5.1.2 Gauss-Seidel Method

The Gauss–Seidel method, especially the related SOR method was one of the most popular methods in solving linear algebraic system arising from the discretization of partial differential equations, c.f. Young [89]. As a single algorithm itself, it is probably not as competitive as the multigrid method in many applications, but it still plays an important role in multigrid algorithms, and is often employed as a smoother.

Algorithm

In the discussion of the Gauss-Seidel method, we will confine ourselves to the case that A is the matrix in (5.1). Splitting the matrix in the form:

$$A = D - L - U$$

where D is the diagonal of A and $-L$ and $-U$ are the strictly lower and upper triangular parts of A, respectively. the Gauss-Seidel method for the system (5.1) can be obtained in (5.2) by choosing

$$S = (D - L)^{-1}.$$

In this case, we have

$$K = I - (D - L)^{-1} A$$

and

$$K^* = I - (D - U)^{-1} A.$$

A direct manipulation shows that

$$I - K^* K = (D - U)^{-1} D (D - L)^{-1} A \tag{5.5}$$

which immediately implies that $I - K^* K$ is positive, thus $\rho(K) < 1$. This gives a proof of a well-known fact that the Gauss-Seidel method converges for any positive definite system (5.1).

Application to Finite Element Equations

Assume $A = A^k$ is the stiffness matrix as described in Section 3.3 for quasiuniform triangulations, namely

$$A = (A(\bar{\phi}_i, \bar{\phi}_j))_{n \times n}$$

where $n = n_k$ and $\{\bar{\phi}_i\}$ is the scaled nodal basis. Notice that in this case $\rho(A) \asymp 1$, hence the following result verifies (5.4).

Proposition 5.1

$$|A\xi|^2 \lesssim (A\xi, (I - K^* K)\xi), \quad \forall \xi \in \mathbb{R}^n$$

where $(\cdot, \cdot)$ *is the Euclidean inner product and* $|\cdot|$ *is the corresponding norm.*

Proof. By (5.5), it suffices to show that

$$|(D - L)\xi|^2 \lesssim (D\xi, \xi), \quad \forall \xi \in \mathbb{R}^n. \tag{5.6}$$

It follows from the quasiuniformity that

$$(D\xi, \xi) \asymp |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Hence (5.6) is equivalent to the following:

$$\sum_{i=1}^{n} |\sum_{j>i} A(\bar{\phi}_i, \bar{\phi}_j)\xi_j|^2 \lesssim \sum_{i=1}^{n} |\xi_i|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Notice that for any given i

$$A(\bar{\phi}_i, \bar{\phi}_j) = 0, \quad \text{if } S_j \cap S_i = \emptyset$$

where $S_i = \text{supp } \phi_i$. But there are only a fixed number of S_j's such that $S_j \cap S_i \neq \emptyset$. Hence, by Schwarz' inequality,

$$\sum_{i=1}^{n} |\sum_{j>i} A(\bar{\phi}_i, \bar{\phi}_j)\xi_j|^2$$

$$\lesssim \sum_{i=1}^{n} |\sum_{S_j \cap S_i \neq \emptyset} |A(\bar{\phi}_i, \bar{\phi}_j)|^2 \sum_{S_j \cap S_i \neq \emptyset} |\xi_j|^2$$

$$\lesssim \sum_{i=1}^{n} \xi_i^2.$$

This completes the proof. ■

5.2 Some Facts about Preconditioners

In Section 3.13 of Chapter 3, for a SPD operator A on a finite dimensional vector space $\mathcal{M}$, we have defined the condition number of A to be the ratio of its maximum eigenvalue to its minimum eigenvalue. This definition also extends to the product of two SPD operators. Namely if B is another SPD operator on $\mathcal{M}$, we then define the condition number of BA by

$$\kappa(BA) = \frac{\lambda_{\max}(BA)}{\lambda_{\min}(BA)},$$

where $\lambda_{\max}(BA)$ and $\lambda_{\min}(BA)$ are the maximum and minimum eigenvalues of BA respectively.

For a given SPD operator A on $\mathcal{M}$, again we are interested in solving the following equation on $\mathcal{M}$:

$$Ax = b. \tag{5.7}$$

It is well–known that the condition number $\kappa(A)$ is very crucial to many algorithms for solving (5.7), a typical example is the conjugate gradient method we will discuss in the next section. Namely the smaller is the condition number, the more efficient is the given algorithm.

The central idea in the preconditioning method is somehow to reduce the condition number of the system (5.7). The standard approach is to find another SPD operator B on $\mathcal{M}$ so that the equivalent system

$$BAx = Bb$$

is better conditioned, namely $\kappa(BA)$ is relatively smaller. The operator B in the above is often called a preconditioner. Obviously a good preconditioner needs to have the following two properties:

1. The action of B is easy to compute.

2. $\kappa(BA)$ is small.

How "easy" or how "small" is relative and depends on the background of the underlying problem. We will discuss this issue in the concrete applications.

To estimate the condition number $\kappa(BA)$ often centers the theoretical analysis of a preconditioning method. The following simple result is most fundamental.

Lemma 5.1 *If μ_0 and μ_1 are positive constants satisfying*

$$\mu_0(Au, u) \le (B^{-1}u, u) \le \mu_1(Au, u), \qquad \forall u \in \mathcal{M}. \tag{5.8}$$

Then

$$\kappa(BA) \le \frac{\mu_1}{\mu_0}.$$

Proof. By changing variable $u = A^{-\frac{1}{2}}v$ in (5.8), we see that (5.8) is equivalent to

$$\sigma(A^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}}) \subset (\mu_0, \mu_1). \tag{5.9}$$

But

$$A^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}} = A^{\frac{1}{2}}(BA)^{-1}A^{-\frac{1}{2}}.$$

Hence

$$\sigma((BA)^{-1}) \subset (\mu_0, \mu_1).$$

Thus

$$\sigma(BA) \subset (\mu_1^{-1}, \mu_0^{-1}),$$

and consequently

$$\kappa(BA) \le \frac{\mu_0^{-1}}{\mu_1^{-1}} = \frac{\mu_1}{\mu_0}.$$

■

Frequently, it is not convenient to verify the inequality of (5.8) that measures the preconditioner. In the following we state some almost trivial but very important equivalent conditions.

Theorem 5.1 *Assume A, B are symmetric positive definite operators on a finite dimensional space $\mathcal{M}$.* (5.8) *is equivalent to either of the following:*

$$\mu_0(Bu,u) \le (A^{-1}u,u) \le \mu_1(Bu,u), \qquad \forall u \in \mathcal{M}. \tag{5.10}$$

$$\mu_1^{-1}(Au,u) \le (ABAu,u) \le \mu_0^{-1}(Au,u), \qquad \forall u \in \mathcal{M}. \tag{5.11}$$

Before ending this section, we include the following simple result

Proposition 5.2 *Assume that A and B are SPD operators on $\mathcal{M}$ and $\sigma > 0$ and $\delta \in (0,1)$ are constants. Then the following are equivalent:*

1. $-\sigma(Au,u) \le (A(I-BA)u,u) \le \delta(Au,u), \quad \forall u \in \mathcal{M}.$

2. $(1-\delta)(Au,u) \le (B^{-1}u,u) \le (1+\sigma)(Au,u), \qquad \forall u \in \mathcal{M}.$

The proof of above result is just a direct algebraic manipulation. If $\sigma \in (0,1)$, the above result means that if $I - BA$ is a contraction. This fact will be used in the multigrid algorithms to be developed.

5.3 Conjugate Gradient Method

The conjugate gradient method is no doubt one of the most fundamental methods in modern approaches to solving linear algebraic systems. Many algorithms presented in this work are based on a variation of this method, namely the preconditioned conjugate gradient method. Because of its extraordinary importance, we will give a rather complete description of this method in this section. The material here was mostly taken from the unpublished notes of Bramble on preconditioning methods.

5.3.1 Definition and error analysis of the algorithm

Let $A : \mathcal{M} \mapsto \mathcal{M}$ be a SPD operator with respect to the inner product $(\cdot, \cdot)$. We need to solve the equation of the following type:

$$Au = f. \tag{5.12}$$

Let n =dim$\mathcal{M}$, it follows from the Cayly–Hamilton Theorem that

$$A^{-1} = p_{n-1}(A),$$

for some polynomial p_{n-1} of the degrre $\leq n-1$. Thus

$$u = A^{-1}f = p_{n-1}(A)f,$$

which means that

$$u \in V_n,$$

where V_k, for $k = 1, 2, \cdots, n$, is defined by

$$V_k = \text{Span}\{f, Af, \cdots, A^{k-1}f\}.$$

The conjugate gradient method can be viewed as the Galerkin projection of u onto V_k defined by $u_k \in V_k$ such that

$$(Au_k, v) = (f, v) \quad \forall \in V_k. \tag{5.13}$$

From our earlier discussion, we see that

$$u_n = u$$

This says that the algorithm is a direct method. But the most important property of this algorithm is that it can also be regarded as an iterative scheme as we shall see later.

An advantage of looking at the conjugate gradient method in the way described above is that the analysis of the error reduction is very easy.

Theorem 5.2

$$(A(u-u_k), u-u_k) \le 4\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^{2k}(Au,u)$$

where $\kappa(A)$ is the condition number of A.

Proof. It is routine to verify that

$$(A(u-u_k), u-u_k) \le (A(u-v), u-v) \quad \forall v \in V_k.$$

Now for an arbitrary polynomial p_{k-1} of degree $k-1$, taking $v = p_{k-1}(A)f = Ap_{k-1}(A)u$, we get

$$\begin{aligned}(A(u-u_k), u-u_k) &\le \min_{p_{k-1}}(A(I-Ap_{k-1}(A))f, (I-Ap_{k-1}(A))u)\\ &\le \min_{q_k(0)=1}(Aq_k(A)u, q_k(A)u)\\ &\le \min_{q_k(0)=1}\max_{\lambda\in\sigma(A)}|q_k(\lambda)|^2(Au,u)\end{aligned}$$

It is well-known that the solution of this min-max problem is in terms of the Chebychev polynomials. We can then deduce that

$$\min_{q_k(0)=1}\max_{\lambda\in\sigma(A)}|q_k(\lambda)| \le 2\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^{k}$$

The desired result then follows. ■

We notice that this estimate for the convergence rate of the conjugate gradient method strongly depends on the condition number. When $\kappa(A)$ is too large, the convergence could be very slow. PCG is just designed to reduce the magnitude of the original condition number and hence improve the convergence rate. This will be discussed later on.

5.3.2 Implementation of the algorithm

The definition of the CG method in (5.13) does not give us much idea how the algorithm can be efficiently implemented and it is also not clear if it can be implemented in an iterative fashion. A little bit more work is needed to find the right (well–known) formulae. Now we are going to show that the computation of CG solution can be reduced to a simple iterative procedure.

To begin with, set $u_0 = 0, r_0 = f, r_j = f - Au_j$ and define for $k = 1, 2, \dots,$

$$V_k^{\perp} = \{w \in V_{k+1} : (Aw, v) = 0, \quad \forall v \in V_k\}$$

1. If $r_k = 0$, then $u_k = u$: we are done. Otherwise $r_k \neq 0$, and then, because $(r_k, v) = (f - Au_k, v) = 0, \quad \forall v \in V_k$, it follows that $r_k \notin V_k$. Hence $V_k^{\perp} \neq 0$ since $r_k \in V_{k+1},$. Thus

$$V_{k+1} = V_k \oplus V_k^{\perp}. \tag{5.14}$$

2. It follows from (5.14) that for $p_k \in V_k^{\perp}, p_k \neq 0$, the $k+1$st CG solution can be represented as

$$u_{k+1} = u_k + \alpha_k p_k,$$

where $\alpha_k = (r_k, p_k)/(Ap_k, p_k)$ since by (5.14) it is straightforward to check that

$$(Au_{k+1}, v) = (f, v), \quad \forall v \in V_{k+1}.$$

3. It remains to find some $p_k \in V_k^{\perp}, p_k \neq 0$. To do this we take $p_0 = r_0$ and assume that we have $\{p_0, \ldots, p_{k-1}\}$, clearly

$$(Ap_j, p_l) = 0, \quad \text{if } j \neq l.$$

Now since we have assumed that $r_k \neq 0$, $r_k \notin V_k$. Hence, since $r_k = f - Au_k \in V_{k+1}$, it has a nonzero component in $V_k^{\perp}$. Therefore we can choose p_k to be the A-orthogonal projection of r_k onto $V_k^{\perp}$. But $\{p_0, \ldots, p_{k-1}\}$ clearly forms a basis for V_k, and we get

$$p_k = r_k - \sum_{j=0}^{k-1} \frac{(Ap_j, r_k)}{(Ap_j, p_j)} p_j.$$

But, since $Ap_j \in V_k$ for $j \leq k-2$,

$$(Ap_j, r_k) = (Ap_j, f - Au_k) = 0, \quad \forall j < k-1.$$

Thus

$$p_k = r_k - \frac{(Ap_{k-1}, r_k)}{(Ap_{k-1}, p_{k-1})} p_{k-1}.$$

We summarize the above arguments and formulate the CG algorithm as follows:

CG Algorithm

1. u_0 *arbitrary,* $p_0 = r_0 = f - Au_0$.

2. For $k = 0, 1, \ldots$

$$\begin{aligned} u_{k+1} &= u_k + \alpha_k p_k \\ p_{k+1} &= r_{k+1} - \beta_k p_k \end{aligned}$$

with

$$r_k = f - Au_k,$$

$$\alpha_k = \frac{(r_k, p_k)}{(Ap_k, p_k)},$$

$$\beta_k = \frac{(Ap_k, r_{k+1})}{(Ap_k, p_k)}.$$

5.3.3 Preconditioned conjugate gradient method

Taking a preconditioner B as described before and applying to both hand side of (5.12), we get an equivalent equation as follows:

$$\hat{A}u = \hat{f} \tag{5.15}$$

where $\hat{A} = BA, \hat{f} = Bf$.

Evidently $\hat{A}$ is not symmetric any more with respect to the original inner product, but it is indeed symmetric with respect to a new inner product $(\cdot,\cdot)_A \overset{\text{def}}{=} (A\cdot,\cdot)$ since

$$(\hat{A}u, v)_A = (ABAu, v) = (u, \hat{A}v)_A.$$

Therefore with the new inner product $(\cdot,\cdot)_A$, we can formulate the conjugate gradient method for (5.12) in the same way as we showed before.

It follows from Theorem 5.1 that $\kappa(\hat{A}) \equiv \kappa(BA)$, hence the corresponding convergence estimate is

$$(BA(u-u_k), A(u-u_k)) \leq 2\left(\frac{\sqrt{\kappa(BA)}-1}{\sqrt{\kappa(BA)}+1}\right)^k (BAu, Au)$$

Correspondingly we have the PCG algorithm as follows:

PCG Algorithm

1. u_0 arbitrary, $p_0 = r_0 = Bf - BAu_0$.

2. For $k = 0, 1, \ldots$

$$\begin{aligned} u_{k+1} &= u_k + \alpha_k p_k, \\ p_{k+1} &= r_{k+1} - \beta_k p_k, \end{aligned}$$

with

$$\begin{aligned} r_k &= Bf - BAu_k, \\ \alpha_k &= \frac{(Ar_k, p_k)}{(BAp_k, Ap_k)}, \\ \beta_k &= \frac{(BAp_k, Ar_{k+1})}{(BAp_k, Ap_k)}. \end{aligned}$$

5.4 A Remark

We would like to point out that the preconditioner B we defined above for A in this section is the one whose spectrum is "close" to that of A^{-1}, not of A itself. As we can see from the expressions in PCG algorithm, we only need to compute the action of B (also of A of course) and we do not have to have any information of B^{-1} at all.

In some applications the action of B and B^{-1} are both relatively easy to compute. In such case, the new inner product(we used $(A\cdot, \cdot)$) may be alternatively taken to be $(B^{-1}\cdot, \cdot)$. But we are not interested in such treatment because of the features of the multigrid methods which we shall consider throughout this thesis.

Chapter 6

Some Technical Lemmas

As we discussed in the introductory chapter, some part of the convergence proof for a multigrid algorithm is in the nature of pure algebraic manipulations. Having observed that many such algebraic arguments are quite similar, we abstractly summarize them into a technical lemma, namely Lemma 6.1 below in Section 6.1. The second section of this chapter is devoted to setting up machinery that can be utilized to analyze some multigrid algorithms without using elliptic regularity assumptions.

6.1 Multigrid Lemma

Our first technical lemma, originated from Bramble and Pasciak [21], represents some sort of common algebraic manipulations in the multigrid literature, which will be exploited in estimating the convergence rate of multigrid algorithms in Chapter 7 and Chapter 9. Since it is the main technical tool for analyzing the convergence of a multigrid algorithm, we call it the *Multigrid Lemma.*

Lemma 6.1 *Assume $\mathcal{M}$ is a finite dimensional vector space with an inner product $(\cdot,\cdot)$. $K \in \mathcal{L}(\mathcal{M},\mathcal{M})$ is a self adjoint operator such that $\sigma(K) \subset [0,1]$ and $\beta \in (0,1]$ amd $C > 0$ are given constants. Then we have following conclusions:*

1. *There exist constants $M, \epsilon_0 > 0$ depending on C and β such that for $\epsilon \in [0,\epsilon_0]$,*

the following estimate holds

$$C(1-\delta^2)((I-K)K^m u, K^m u)^\beta (K^m u, K^m u)^{1-\beta}$$
$$+(1+\epsilon)\delta^2(K^m u, K^m u) \le \delta(u,u) \qquad \forall u \in \mathcal{M} \tag{6.1}$$

with

$$\delta = \frac{1}{(1+m/M)^\beta}. \tag{6.2}$$

2. *Given constants* $\gamma_1 \ge \gamma_0 > 1$*, there exist constants* $M, \epsilon_0 > 0$ *depending on* C, β, γ_0 *and* γ_1 *such that for any* $\epsilon \in [0, \epsilon_0]$ *and* $\gamma \in [\gamma_0, \gamma_1]$*, the following estimate holds*

$$C(1-\delta_0)((I-K)K^m u, K^m u)^\beta (K^m u, K^m u)^{1-\beta}$$
$$+(1+\epsilon)\delta_0(K^m u, K^m u) \le \delta(u,u) \qquad \forall u \in \mathcal{M} \tag{6.3}$$

with

$$\delta_0 = \frac{1}{(\gamma m)^\beta + M}, \tag{6.4}$$

$$\delta = \frac{1}{m^\beta + M}. \tag{6.5}$$

3. *There exist constants* $M, \epsilon_0 > 0$ *depending on* C, β *such that for any integer* $k \ge 1$ *and* $\epsilon \in [0, \epsilon_0]$*,* (6.3) *holds with*

$$\delta_0 = \frac{(k-1)^{1/\beta-1}M}{m^\beta + (k-1)^{1/\beta-1}M}, \tag{6.6}$$

$$\delta = \frac{k^{1/\beta-1}M}{m^\beta + k^{1/\beta-1}M}. \tag{6.7}$$

In the later applications of this lemma, Cases 1, 2 and 3 correspond to the W-cycle, variable V-cycle and V-cycle respectively. The results with $\epsilon = 0$ will be used in the next chapter for SPD problems, whereas the results with $\epsilon-$perturbation will be used in Chapter 9 for nonsymmetric or indefinite problems.

Proof. The proof given here is a modification of proofs of Theorem 1, Theorem 3 and Theorem 5 in [21] where ϵ does not appear (namely $\epsilon = 0$.) Since the modification is quite similar in each of the three cases in the lemma, it seems more than enough to present proofs only for the first two cases; hence the proof for case 3 is omitted.

To begin with, we apply the generalized arithmetic-geometric mean inequality and get for any positive η that

$$\begin{aligned}&((I-K)K^m u, K^m u)^\beta (K^m u, K^m u)^{1-\beta}\\ &\quad \le \beta\eta((I-K)K^m u, K^m u) + (1-\beta)\eta^{-\beta/(1-\beta)}(K^m u, K^m u). \qquad (6.8)\end{aligned}$$

Since K is self-adjoint and its spectrum is contained in $[0,1]$ by hypothesis,

$$\begin{aligned}((I-K)K^m u, K^m u) &\le \frac{1}{2m}\sum_{i=0}^{2m-1}((I-K)K^i u, u)\\ &= \frac{1}{2m}((I-K^{2m})u, u). \qquad (6.9)\end{aligned}$$

For simplicity, denote the left hand side of (6.1) by LFH, then combining (6.8) with (6.9), we get

$$\begin{aligned}LFH &\le [(1-\delta^2)C(1-\beta)\eta^{\frac{-\beta}{1-\beta}} + (1+\epsilon)\delta^2](K^{2m}u, u)\\ &+ (1-\delta^2)C\frac{\beta}{2m}\eta((I-K^{2m})u, u)\end{aligned}$$

holds for any positive η. We define η by the equation

$$(1-\delta^2)C\frac{\beta}{2m}\eta = \delta. \qquad (6.10)$$

It then suffices to show that for η defined by (6.10),

$$(1-\delta_2)C(1-\beta)\eta^{\frac{-\beta}{1-\beta}} + \epsilon\delta^2 \leq \delta - \delta^2, \tag{6.11}$$

Solving for η and using this result in (6.11) implies that it is sufficient to choose M and ϵ_0 so that

$$C_3(1-\delta^2)^{1/(1-\beta)} m^{-\beta/(1-\beta)} + \epsilon\delta^{(2-\beta)/(1-\beta)}$$
$$\leq (1-\delta)\delta^{1/(1-\beta)}.$$

It is elementary to see that

$$2^{-1/(1-\beta)} \leq (1-\delta)^{-\beta/(1-\beta)} \left(\frac{m}{M}\right)^{\beta/(1-\beta)} \left(\frac{\delta}{1+\delta}\right)^{1/(1-\beta)}$$

for δ given by (6.2). Define M by

$$M^{\beta/(2-\beta)} 2^{-2/(2-\beta)} = 2C_3.$$

Then

$$C_3(1-\delta^2)^{1/(1-\beta)} m^{-\beta/(1-\beta)} \leq \frac{1}{2}(1-\delta)\delta^{1/(1-\beta)}.$$

Choosing

$$\epsilon_0 \leq \frac{1-\delta}{2\delta}$$

implies

$$\epsilon\delta^{(2-\beta)/(1-\beta)} \leq \frac{1}{2}(1-\delta)\delta^{1/(1-\beta)}.$$

This completes the proof of conclusion 1.

Next we show (6.3). Following the argument given above, we see that it suffices to show that

$$(1-\delta_0)C(1-\beta)\gamma^{-\alpha/(1-\alpha)} + \epsilon_0\delta_0 \leq \delta - \delta_0, \tag{6.12}$$

holds with δ_0 and δ defined by (6.4) and (6.5), where η is defined by

$$(1-\delta_0)C\frac{\beta}{2m}\eta = \delta_0. \tag{6.13}$$

A direct computation using (6.4) and (6.5) shows that (6.3) is equivalent to

$$\begin{aligned} & C_3\gamma^{\beta/(1-\beta)}M^{-\beta/(1-\beta)} + \epsilon M \\ \leq & \frac{Mm^\beta}{D(k)}(\gamma^\beta - 1). \end{aligned}$$

Note that if $M \geq 1$ then

$$C_4 \equiv (\gamma_0^\beta - 1)/2 \leq \frac{Mm^\beta}{M+m^\beta}(\gamma^\beta - 1).$$

Hence it suffices to have

$$C_5 M^{-\beta/(1-\beta)} + \epsilon M \leq C_4$$

where $C_5 = C_3\gamma_1^{\beta/(2-\beta)}$. Thus taking $M \geq 1$, large enough so that

$$C_5 M^{-\beta/(2-\beta)} \leq C_4/2$$

and

$$\epsilon \leq \epsilon_0 \leq C_4^{1/\eta}(2M)^{-1}$$

completes the proof of the theorem. ■

6.2 A Trick for Analyzing the Two Level Scheme

What is to be presented next is a technique that can be applied to study the two-level convergence behavior of multigrid algorithms when the elliptic regularity assumption is not easy to verify. Applications can be found in Chapter 8 for interface problems, nonuniform grids and nonconforming elements.

Assume we are given two finite dimensional spaces

$$\mathcal{M}_{k-1} \subset \mathcal{M}_k$$

and a SPD bilinear form $A(\cdot,\cdot)$ defined on $\mathcal{M}_k$. Each $\mathcal{M}_l$ is equipped with an inner product $(\cdot,\cdot)_l$ with $l = k-1, k$. $P_{k-1} : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ is the standard Galerkin projection satisfying

$$A(P_{k-1}u, v) = A(u, v), \quad \forall u \in \mathcal{M}_k, v \in \mathcal{M}_{k-1}.$$

For $l = k-1, k$, the operator $A_l : \mathcal{M}_l \mapsto \mathcal{M}_l$ is defined by

$$(A_l u, v)_l = A(u, v), \quad \forall u, v \in \mathcal{M}_l.$$

We make the following principal assumption:

(A6.1) $\quad \inf_{\chi \in \mathcal{M}_{k-1}} \|v - \chi\|_k{}^2 \le C_1 \lambda_k{}^{-1} A(v, v) \quad \forall v \in \mathcal{M}_k,$

where $\lambda_k = \rho(A_k)$ and C_1 is a positive constant.

Lemma 6.2 [1] *Assume $K_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ is a SPD operator satisfying*

$$C_0 \|A_k v\|_k{}^2 \le \lambda_k A((I - K_k)v, v), \quad \forall v \in \mathcal{M}_k. \tag{6.14}$$

*Then, under the assumption (**A6.1**), we have*

$$\begin{aligned} A((I - P_{k-1})K_k^m u, K_k^m u) \qquad &\le (1 - \tfrac{C_0}{C_1}) A(K_k^{2m-1} u, u) \\ &\le (1 - \tfrac{C_0}{C_1}) A(u, u), \quad \forall u \in \mathcal{M}_k. \end{aligned}$$

Proof. By the definition of P_{k-1}, for any $v \in \mathcal{M}_k$ and $\chi \in \mathcal{M}_{k-1}$, we have

$$\begin{aligned} A((I - P_{k-1})v, v) &= A((I - P_{k-1})v, (I - P_{k-1})v - \chi) \\ &= (A_k (I - P_{k-1})v, (I - P_{k-1})v - \chi)_k \\ &\le \|A_k (I - P_{k-1})v\|_k \|(I - P_{k-1})v - \chi\|_k. \end{aligned}$$

[1] Before this thesis was about to be filed, Mandel pointed out to the author that results similar to this lemma were contained in some earlier literature, c.f. Mandel [63], Hackbusch [48], Brandt [31], Kočava and Mandel [56].

Applying (**A6.1**) yields

$$A((I - P_{k-1})v, v) \le C_1 {\lambda_k}^{-1} {\|A_k(I - P_{k-1})v\|_k}^2.$$

Using the hypothesis for K_k, we get that

$$\begin{aligned} & A(K_k(I - P_{k-1})v, (I - P_{k-1})v) \\ \le\ & A((I - P_{k-1})v, v) - C_0 {\lambda_k}^{-1} {\|A_k(I - P_{k-1})v\|_k}^2 \\ \le\ & (1 - \frac{C_0}{C_1}) A((I - P_{k-1})v, v). \end{aligned}$$

Using Schwarz inequality and the above estimate with $v = K_k^m u$, we deduce that

$$\begin{aligned} & A((I - P_{k-1})K_k^m u, K_k^m u)^2 \\ \le\ & A(K_k(I - P_{k-1})K_k^m u, K_k^m u) A(K_k^{2m-1} u, u) \\ \le\ & (1 - \frac{C_0}{C_1}) A((I - P_{k-1})K_k^m u, K_k^m u) A(K_k^{2m-1} u, u). \end{aligned}$$

The desired result then follows. ■

An interesting fact that can be reduced from the above result is that the constants C_0 and C_1 appearing in (**A5.1**) and (6.14) must obey the following relation:

$$C_0 < C_1.$$

Before we present our last result in this chapter, we introduce the following notation:

$$\|E\|_A \stackrel{\text{def}}{=} \sup_{v \in \mathcal{M}_\ell} \frac{A(Ev, v)}{A(v, v)},$$

if $E : \mathcal{M}_\ell \mapsto \mathcal{M}_\ell$ is a nonnegative self adjoint operator.

As a consequence of Lemma 6.2, we have

Lemma 6.3 *Assume $E_l : \mathcal{M}_l \mapsto \mathcal{M}_l$, for $l = k-1, k$ are nonnegative self adjoint operators that are related by*

$$E_k = K_k^m (I - P_{k-1} + E_{k-1}^p P_{k-1}) K_k^m,$$

where $p \geq 1$ is an integer. Then, under the assumptions of Lemma 6.2

$$\| E_k \|_A \leq (1-\eta) \| E_{k-1} \|_A^p + \eta,$$

where $\eta = 1 - \frac{C_0}{C_1}$. Hence $\| E_{k-1} \|_A < 1$ implies that $\| E_k \|_A < 1$.

Proof. Denote $\delta_{k-1} = \| E_{k-1} \|_A$. It is routine to see that

$$\begin{aligned} A(E_k u, u) &\leq (1-\delta_{k-1}^p) A((I - P_{k-1}) K_k^m u, K_k^m u) + \delta_{k-1}^p A(K_k^m u, K_k^m u) \\ &\leq ((1-\delta_{k-1}^p)\eta + \delta_{k-1}^p) A(K_k^{2m-1} u, u) \\ &\leq ((1-\eta)\delta_{k-1}^p + \eta) A(K_k^{2m-1} u, u) \\ &\leq ((1-\eta)\delta_{k-1}^p + \eta) A(u, u). \end{aligned}$$

■

Chapter 7

An Abstract Multigrid Theory for SPD Problems

The aim of this chapter is to establish a more general framework of the multigrid method for symmetric positive definite problems. Applications can be found in the next chapter for finite element discretizations for partial differential equations. In particular, a theory for curved boundary domain problems is established by means of the general framework in this chapter.

The outline of this chapter is as follows. In Section 7.1, we set up an abstract framework in which a symmetric multigrid algorithm is given. Based on various assumptions that are described in Section 7.2, a number of convergence theorems are proven in Section 7.3. Section 7.4 is concerned with a kind of algebraic multigrid algorithm.

7.1 Framework and Algorithm

Assume we are given a Hilbert space H and a hierarchy of real finite dimensional subspaces of H

$$\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_j$$

which are related by the so-called prolongation operators $\mathcal{I}_k : \mathcal{M}_{k-1} \mapsto \mathcal{M}_k$.

In addition, let $A_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ be symmetric positive definite bilinear forms on $\mathcal{M}_k$. We shall develop multigrid algorithms for the solution of the following problem: Given $f \in \mathcal{M}_j$, find $u \in \mathcal{M}_j$ satisfying

$$A_j(u,\phi) = (f,\phi)_j \quad \forall \phi \in \mathcal{M}_j.$$

To define the multigrid algorithms, we need to define some auxiliary operators. For $k = 1, \ldots, j$, the operator $A_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ is defined by

$$(A_k w, \phi)_k = A_k(w,\phi) \quad \forall w, \phi \in \mathcal{M}_k.$$

Clearly the operator A_k is symmetric positive definite (in both the $A_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ inner products). In terms of the prolongation operator $\mathcal{I}_k$, we have operators $\mathcal{I}_k^t : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ and $\mathcal{I}_k^* : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ defined by

by

$$(\mathcal{I}_k^t w, \phi)_{k-1} = (w, \mathcal{I}_k \phi)_k \quad \forall w \in \mathcal{M}_k, \phi \in \mathcal{M}_{k-1}. \tag{7.1}$$

and

$$A_{k-1}(\mathcal{I}_k^* w, \phi) = A_k(w, \mathcal{I}_k \phi) \quad \forall w \in \mathcal{M}_k, \phi \in \mathcal{M}_{k-1}. \tag{7.2}$$

In other words, $\mathcal{I}_k^t$ and $\mathcal{I}_k^*$ are the adjoints of $\mathcal{I}_k$ with the inner products $(\cdot,\cdot)_k$ and $A_k(\cdot,\cdot)$ respectively. It is straightforward to check that

$$\mathcal{I}_k^* = A_{k-1}^{-1} \mathcal{I}_k A_k. \tag{7.3}$$

$\mathcal{I}_k^t$ is often called *restriction* operator, which is another main ingredient of any multigrid algorithm. $\mathcal{I}_k^*$, which is related the Galerkin projection in some applications, will only be used in the analysis.

Another important component of the multigrid algorithm is called *smoothing*, which will be represented by a sequence of linear operators $S_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ for $1 \le k \le j$ to define the smoothing process. These operators may be symmetric or nonsymmetric with respect to the inner product $(\cdot,\cdot)_k$. If S_k is not symmetric, then we denote by S_k^t its adjoint and set

$$S_k^{(l)} = \begin{cases} S_k & \text{if } l \text{ is odd;} \\ S_k^t & \text{if } l \text{ is even.} \end{cases}$$

With the framework and notation given above, we are now in a position to define our multigrid algorithm, which will be characterized in terms of a sequence of recursively defined operators $B_k : \mathcal{M}_k \mapsto \mathcal{M}_k$. In the following, p, m_k are given positive integers and λ_k is either equal to $\rho(A_k)$ or an upper bound of $\rho(A_k)$ such that $\lambda_k \asymp \rho(A_k)$.

Algorithm S

Step 1 $B_1 = A_1^{-1}$.

Step 2 *Assume B_{k-1} is defined. Then B_k is defined, for $g \in \mathcal{M}_k(A_k w = g)$, as follows:*

1. *Pre-smoothing on $\mathcal{M}_k$:*

$$\begin{aligned} & w^0 = 0 \\ & w^l = w^{l-1} + S_k^{(l+m_k)}(g - A_k w^{l-1}) \\ & l = 1, 2, \cdots, m_k. \end{aligned}$$

2. *Correction on $\mathcal{M}_{k-1}$: $w^{m_k+1} = w^{m_k} + \mathcal{I}_k q^p$ where $q^p \in \mathcal{M}_{k-1}$ is defined as follows*

$$\begin{aligned} & q^0 = 0 \\ & q^l = (I - B_{k-1}A_{k-1})q^{l-1} + B_{k-1}\mathcal{I}_k^t(g - A_k w^{m_k}) \\ & l = 1, 2, \cdots, p. \end{aligned}$$

3. *Post-smoothing on $\mathcal{M}_k$:*

$$\begin{aligned} & w^l = w^{l-1} + S_k^{(l+m_k+1)}(g - A_k w^{l-1}) \\ & l = m_k + 2, \cdots, 2m_k + 1. \end{aligned}$$

Define: $B_k g = w^{2m_k+1}$.

Remark Ordinary multigrid algorithms can be more general than what we have given above. For example, In Step 1, B_1 may be defined by an iterative method which solves the equation approximately on $\mathcal{M}_1$. Another generalization is that the number of pre- and post-smoothings are not necessarily the same. Nevertheless we are not going to consider these more general cases in this dissertation. However in some circumstances it seems crucial to our theory that the number of pre- and post smoothings should be the same, which will guarantee that the multigrid operators B_k are also symmetric(see Lemma 7.2). This is reasonable and important from many viewpoints. The most natural reason would be because the original problems are symmetric themselves.

A great advantage in setting out the algorithm by means of the operators B_k is that we have a very simple recurrence relation for the "residue"operator $E_k \stackrel{\text{def}}{=} I - B_k A_k$ as given in the following lemma.

Lemma 7.1 *Let $E_k = I - B_k A_k$ and $K_k = I - S_k A_k$. Then*

$$E_k = (\widetilde{K}_k^{m_k})^* \left((I - \mathcal{I}_k \mathcal{I}_k^*) + \mathcal{I}_k E_{k-1}^p \mathcal{I}_k^* \right) \widetilde{K}_k^{m_k}. \tag{7.4}$$

Furthermore, for any $u, v \in \mathcal{M}_k$

$$A_k(E_k u, v) = A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\tilde{u}, \tilde{v}) + A_{k-1}(E_{k-1}^p \mathcal{I}_k^* \tilde{u}, \mathcal{I}_k^* \tilde{v}), \tag{7.5}$$

where $\tilde{u} = \widetilde{K}_k^{m_k} u$ and

$$\widetilde{K}_k^{m_k} = \begin{cases} (K_k^* K_k)^{\frac{m-1}{2}} K_k^*, & \textit{if } m \textit{ is odd;} \\ (K_k^* K_k)^{\frac{m}{2}}, & \textit{if } m \textit{ is even.} \end{cases}$$

The verification of the above lemma is straightforward by the definition of the algorithm. The next thing we want to address is that the **Algorithm S** defines a symmetric operator. More specifically, we have

Lemma 7.2 *B_k is symmetric with respect to $(\cdot,\cdot)_k$and E_k is symmetric with respect to $A_k(\cdot,\cdot)$.*

Proof. The symmetry of E_k follows readily from (7.5) by induction since $E_1 = 0$. The symmetry of B_k with respect to $(\cdot,\cdot)_k$ is evidently implied by the symmetry of E_k with respect to $A_k(\cdot,\cdot)$. ■

We observe that in the algorithm stated above, p and m_k are free parameters. With different parameters, we will take account of the three types of the algorithms named in the following:

Definition 7.1 *The* **Algorithm S** *is known as the*

1. *V-cycle if $p = 1$ and $m_k = m \geq 1$ for $k = 1, \cdots, j$,*
2. *W–cycle if $p = 2$ and $m_k = m \geq 1$ for $k = 1, \cdots, j$,*
3. *Variable V-cycle if $p = 1$ and $\gamma_0 m_k \leq m_{k-1} \leq \gamma_1 m_k$ for $k = 1, \cdots, j$, where γ_0 and γ_1 are constants greater than 1.*

7.2 Assumptions

Our convergence theory will be built upon a number of assumptions described below. As will be shown in the next section, different set of assumptions will lead to different convergence results. All these assumptions will be verified in the context of the finite element methods in the next chapter.

The first assumption, which we have mentioned in Chapter 5(see (5.4)), is on the smoothing operator S_k:

(A7.0) $$\|A_k v\|_k^2 \leq C_0 \lambda_k A_k((I - K_k^* K_k)v, v)$$

where $K_k = I - S_k A_k$.

This condition implicitly means that the smoothing process converges at least as fast as some damped Richardson's method. As a matter of fact, it is equivalent to the following assumption which was used by McCormick in [69] and [70]:

$$A_k(K_k u, K_k u) \le A_k(K_{k,\omega} u, K_{k,\omega} u), \quad \forall u \in \mathcal{M}_k$$

for some $\omega \in (0,2)$ independent of k, where $K_{k,\omega} = I - \omega\lambda_k^{-1} A_k$.

In the finite element applications, (**A7.0**) is easily satisfied if for example, the Richardson or Gauss-Seidel method is used. This was discussed in Chapter 5. We note also that this assumption only involves a single level, hence its verification is relatively easy.

A direct consequence of (**A7.0**) is

$$\textbf{(A7.0}') \qquad \rho(K_k) < 1,$$

where $\rho(\cdot)$ denotes the spectral radius. This assumption will be used in place of (**A7.0**) to get some more general (but weaker) results.

The second assumption, usually called "regularity and approximation assumption", is that there exist a constant $\beta \in (0,1]$ and a constant C_1 such that

$$\textbf{(A7.1)} \qquad |A_k((I - \mathcal{I}_k\mathcal{I}_k^*)v, v)| \le C_1(\lambda_k{}^{-1}\|A_k v\|_k{}^2)^\beta A_k(v,v)^{1-\beta} \quad \forall v \in \mathcal{M}_k.$$

This is the most crucial assumption in our multigrid theory. It relates the bilinear forms, different levels of spaces and prolongation operators. In the case of elliptic boundary value problems, its verification is strongly tied to the regularity property of the underlying partial differential equation.

It is not hard to see that (**A7.1**) implies that

$$A_k(\mathcal{I}_k v, \mathcal{I}_k v) \le \widetilde{C}_1 A_{k-1}(v,v), \quad \forall v \in \mathcal{M}_{k-1}. \tag{7.6}$$

The inequality of this type is also useful in the multigrid theory. With the assumptions (**A7.0**) and (**A7.1**), we will show that the variable V–cycle multigrid algorithm would provide an optimal preconditioner even though it is in general not convergent in the usual sense. In order to get the strongest result, namely that the algorithm actually gives a uniform contraction in each iteration, it is sufficient to assume a stronger version of (7.6):

(A7.2) $$A_k(\mathcal{I}_k v, \mathcal{I}_k v) \leq A_{k-1}(v, v) \quad \forall v \in \mathcal{M}_{k-1}.$$

This assumption guarantees that E_k be nonnegative (c.f. Lemma 7.4). Together with (**A7.0**) and (**A7.1**), as we mentioned before, optimal multigrid convergence results may be obtained. But on the other hand, even without assuming (**A7.0**) and (**A7.1**), this assumption together with (**A**7.0′) is enough to guarantee the convergence of the algorithm if the convergence rate is not a concern.

We will also develop a theory when (**A7.2**) is replaced by the following weaker assumption: For some $\gamma \in (0, 1]$,

(A7.2′**)** $$A_k(\mathcal{I}_k v, \mathcal{I}_k v) \leq (1 + C_2 \lambda_k{}^{-\gamma}) A_{k-1}(v, v), \quad \forall v \in \mathcal{M}_{k-1}.$$

It is possible to scale the forms $A_k(\cdot, \cdot)$ so that (**A7.2**) always holds. But generally this scaling may violate (**A7.1**). However it is not hard to see, when (**A**7.2′) is satisfied, a proper scaling of $A_k(\cdot, \cdot)$ will give rise to (**A7.1**) while (**A7.2**) is still valid. In applications, (**A7.1**) is usually the most difficult assumption to be verified.

Observing that $\mathcal{I}_k^*$ is the adjoint of $\mathcal{I}_k$, we have the following

Lemma 7.3 *(**A7.2**) and (**A**7.2′) are equivalent respectively to the following:*

(a.2) $$A_{k-1}(\mathcal{I}_k^* u, \mathcal{I}_k^* u) \leq A_k(u, u) \quad \forall u \in \mathcal{M}_k,$$

(a.2′**)** $$A_{k-1}(\mathcal{I}_k^* u, \mathcal{I}_k^* u) \leq (1 + C_2 \lambda_k{}^{-\gamma}) A_k(u, u) \quad \forall u \in \mathcal{M}_k.$$

We have observed the symmetry of B_k and E_k in Lemma 7.2. The following lemma is about their positivity property.

Lemma 7.4 *The multigrid operator B_k is positive for the V–cycle or variable V–cycle if the assumption (*$\mathbf{A}7.0'$*) holds. The residue operator E_k is nonnegative if the assumption (***A7.2***) hold.*

Proof. ($\mathbf{A}7.0'$) implies that $I-\bar{K}_k^{m_k}$ is positive. The positivity of B_k then follows by using an induction argument with the following identity from (7.5):

$$(B_kA_ku, A_ku)_k = A_k((I-\bar{K}_k^{m_k})u, u) + (B_{k-1}A_{k-1}\mathcal{I}_k^*\widetilde{K}_k^{m_k}u, A_{k-1}\mathcal{I}_k^*\widetilde{K}_k^{m_k}u)_{k-1}.$$

It follows from Lemma 7.3 that

$$A_k((I-\mathcal{I}_kP_{k-1})u, u) = A_k(u, u) - A_{k-1}(\mathcal{I}_k^*u, \mathcal{I}_k^*u) \geq 0$$

Therefore applying induction to (7.5) shows that E_k is nonnegative, completing the proof. ■

7.3 Convergence Analysis

In this section, we give an analysis of the multigrid algorithm described in the previous section. The first goal of this section is to prove that an inequality of the form

$$-\sigma_kA(v, v) \leq A_k(E_kv, v) \leq \delta_kA(v, v), \quad \forall v \in \mathcal{M}_k \tag{7.7}$$

holds with positive constants $\delta_k < 1$ and σ_k for $k = 1, \cdots, j$.

If $\sigma_k < 1$, then (7.7) gives the standard contraction property of the multigrid algorithm, namely

$$|||E_k|||_k \leq \max(\sigma_k, \delta_k) < 1,$$

where

$$|||E_k|||_k \stackrel{\text{def}}{=} \max_{v\in\mathcal{M}_k} \frac{|A(E_kv, v)|}{A(v, v)}.$$

As we mentioned in Chapter 5 (c.f. Proposition 5.2), (7.7) is equivalent to the following:

$$(1-\delta_k)A_k(v, v) \leq A_k(B_kA_kv, v) \leq (1+\sigma_k)A_k(v, v), \quad \forall v \in \mathcal{M}_k. \tag{7.8}$$

This implies that B_j is a preconditioner of A_j with the condition number

$$\kappa(B_jA_j) \leq \frac{1+\sigma_j}{1-\delta_j}.$$

Hence if we apply multigrid algorithms with the PCG method described in Chapter 5, we only need that δ_j stays away from 1 and σ_j is bounded above.

Our first result is to give an estimate on δ_k in (7.7).

Lemma 7.5 *Under the assumptions (**A7.0**) and (**A7.1**), there exists a constant $M > 0$, depending on C_0, C_1 and β but independent of k, such that*

$$A_k(E_ku, u) \leq \delta_k A_k(u, u), \quad \forall u \in \mathcal{M}_k \tag{7.9}$$

where, for V–cycle

$$\delta_k = \frac{k^{1/\beta-1}M}{m^\beta + k^{1/\beta-1}M}, \tag{7.10}$$

for variable V–cycle

$$\delta_k = \frac{M}{m_k^\beta + M}, \tag{7.11}$$

and for W–cycle

$$\delta_k = \frac{M^\beta}{(m+M)^\beta}. \tag{7.12}$$

Proof. We shall use induction argument. It is trivial for $k = 1$. Assume (7.9) holds for $k-1$, then it follows from (7.5) that

$$\begin{aligned} A_k(E_ku, u) &= A_k((I - \mathcal{I}_k\mathcal{I}_k^*)\tilde{u}, \tilde{u}) + A_{k-1}(E_{k-1}\mathcal{I}_k^*\tilde{u}, \mathcal{I}_k^*\tilde{u}) \\ &\leq A_k((I - \mathcal{I}_k\mathcal{I}_k^*)\tilde{u}, \tilde{u}) + \delta_{k-1}A_{k-1}(\mathcal{I}_k^*\tilde{u}, \mathcal{I}_k^*\tilde{u}) \\ &= (1-\delta_{k-1})A_k((I - \mathcal{I}_k\mathcal{I}_k^*)\tilde{u}, \tilde{u}) + \delta_{k-1}A_k(\tilde{u}, \tilde{u}) \end{aligned}$$

where $\widetilde{u} = \widetilde{K}_k^{(m_k)} u$.

Applying (**A7.0**) yields

$$\frac{\|A_k \widetilde{u}\|_k^{\,2}}{\lambda_k} \le C A_k((I - \bar{K}_k)\bar{K}_k^m u, u)$$

where

$$\bar{K}_k = \begin{cases} K_k K_k^* & \text{if } l \text{ is odd;} \\ K_k^* K_k & \text{if } l \text{ is even.} \end{cases}$$

Combining the above estimates with (**A7.1**), we get

$$\begin{aligned} & A_k(E_k u, u) \\ \le \;& C(1 - \delta_{k-1}^p) A_k((I - \bar{K}_k)\bar{K}_k^{m_k} u, u)^\beta A_k(\bar{K}_k^{m_k} u, u)^{1-\beta} \\ + \;& \delta_{k-1}^p A_k(\bar{K}_k^{m_k} u, u). \end{aligned}$$

At this point, we recognize that the above inequality takes the form of the assumption of our first technical lemma in Chapter 6, hence the desired results are then deduced by applying Lemma 6.1. ■

The above lemma only supplies the estimate for δ_k in (7.7). In order to get the estimate of σ_k, we need some additional assumptions. For example, (7.7) follows directly with $\sigma_k = 0$ if (**A7.2**) is valid, which will be stated separately later. The following lemma is preliminary to getting the final result in some other cases.

Lemma 7.6 (Variable V–cycle) *Assume that (***A7.1***) holds. Then for the variable V-cycle of* **Algorithm S** *, we have*

$$-A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \le \eta_k A_k(u, u) \tag{7.13}$$

where

$$\eta_k \lesssim m_k^{-\beta},$$

or

$$\eta_k \lesssim \min(\lambda_k^{-\gamma}, m_k^{-\gamma}),$$

if ($\mathbf{A}7.2'$) *also holds. As a consequence*

$$-A_k(E_k u, u) \le (\tau_k - 1) A_k(u,u), \quad \forall u \in \mathcal{M}_k, \tag{7.14}$$

where $\tau_k = \prod_{i=1}^k (1+\eta_i)$.

Proof. Using the same argument as in (6.9), we deduce that

$$\lambda_k{}^{-1} A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \le A_k((I-K_k)\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \lesssim \frac{1}{m_k} A_k(u,u). \tag{7.15}$$

Combining (7.15) with (**A7.1**), we get

$$\begin{aligned} -A_k((I-\mathcal{I}_k\mathcal{I}_k^*)\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) &\lesssim (\lambda_k^{-1} \|A_k \tilde{u}\|_k{}^2)^\beta A_k(u,u)^{1-\beta} \\ &\lesssim m_k^{-\beta} A_k(u,u). \end{aligned}$$

Now if ($\mathbf{A}7.2'$) holds, it then follows from Lemma 7.3 that

$$\begin{aligned} &-A_k((I-\mathcal{I}_k\mathcal{I}_k^*)\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \\ &= A_{k-1}(\mathcal{I}_k^*\widetilde{K}_k^{(m_k)} u, \mathcal{I}_k^*\widetilde{K}_k^{(m_k)} u) - A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \\ &\le C_2 \lambda_k{}^{-\gamma} A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u). \end{aligned}$$

On the one hand

$$\lambda_k{}^{-\gamma} A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \le \lambda_k{}^{-\gamma} A_k(u,u),$$

and on the other hand, by (7.15)

$$\begin{aligned} &\lambda_k{}^{-\gamma} A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) \\ &\le (\lambda_k{}^{-1} A_k(\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u))^\gamma A_k(u,u)^{1-\gamma} \\ &\lesssim m_k^{-\gamma} A_k(u,u). \end{aligned}$$

Combining the above two inequalities proves (7.13).

Now we turn to the proof of (7.14). Again we prove it by induction. For $k = 1$ there is nothing to prove. Assume (7.14) holds for $k-1$. Then by (7.4), the induction hypothesis and (7.13),

$$\begin{aligned}
&-A_k(E_k u, u) \\
&\quad = \; -A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\widetilde{K}_k^{(m_k)} u, \widetilde{K}_k^{(m_k)} u) - A_{k-1}(E_{k-1}\mathcal{I}_k^* \widetilde{K}_k^{(m_k)} u, \mathcal{I}_k^* \widetilde{K}_k^{(m_k)} u) \\
&\quad \leq \; \eta_k A_k(u,u) + \sigma_{k-1} A_{k-1}(\mathcal{I}_k^* \widetilde{K}_k^{(m_k)} u, \mathcal{I}_k^* \widetilde{K}_k^{(m_k)} u) \\
&\quad \leq \; [\eta_k + \sigma_{k-1}(1+\eta_k)] A_k(u,u) = \sigma_k A_k(u,u).
\end{aligned}$$

This completes the proof. ■

7.3.1 Estimates when (A7.0), (A7.1) and (A7.2) hold

If all these assumptions are satisfied, the strongest results can be attained. As a direct consequence of Lemma 7.5 and Lemma 7.4, we have

Theorem 7.1 *Assume that (***A7.0***), (***A7.1***) and (***A7.2***) hold. Then* (7.7) *holds with*

$$\sigma_k = 0$$

and with δ_k *given by* (7.10), (7.11) *and* (7.12) *for V–cycle, variable V–cycle and W–cycle respectively.*

In the later applications to finite elements, these types of results hold for nested multigrid algorithms, see Section 8.2. Also certain types of nonnested algorithms (for curved boundary domain), see Section 8.3.

7.3.2 Estimates when (A7.0), (A7.1) and (A$7.2'$) hold

We have pointed out before that the assumption (**A7.2**) is not always satisfied in applications, hence we want to establish some theory that is not contingent upon it. Our first step toward relaxing this assumption is to replace it by ($\mathbf{A}7.2'$). We will only gives results for V–cycle and variable V–cycle.

Theorem 7.2 (Variable V–cycle) *Assume that* (**A7.0**) *and* (**A7.1**) *and* ($\mathbf{A}7.2'$) *hold. Then for the variable V-cycle of* **Algorithm S** , (7.7) *holds with*

$$\sigma_k = \prod_{i=1}^{k}(1+\frac{C_2}{\lambda_i^\gamma}) - 1, \quad \delta_k = \frac{M}{m_k^\beta + M}$$

Proof. Apply Lemma 7.5 and Lemma 7.6. ■

Corollary 7.1 *In addition to the assumptions in Theorem 7.2, we assume that there exists a constant* $\mu > 1$ *such that*

$$\lambda_i \geq \mu^{i-j}\lambda_j, \quad i \leq j. \tag{7.16}$$

Then, if λ_j *is sufficiently large that* $\lambda_j^\gamma \geq \frac{2C_2\mu^\gamma}{\mu^\gamma - 1}$, *we have*

$$\sigma_j \leq \frac{2C_2\mu^\gamma}{\mu^\gamma - 1}\frac{1}{\lambda_j^\gamma} < 1.$$

Consequently

$$\|E_j\|_j \leq \max(\frac{2C_2\mu^\gamma}{\mu^\gamma - 1}\frac{1}{\lambda_j^\gamma}, \frac{M}{m_k^\beta + M}) < 1.$$

Proof. By hypothesis

$$\begin{aligned}
\prod_{i=1}^{j}(1+\frac{C_2}{\lambda_i^\gamma}) &\leq \exp(\sum_{i=1}^{j}\frac{C_2}{\lambda_i^\gamma}) \\
&\leq \exp(\sum_{i=1}^{j}\frac{C_2}{\lambda_i^\gamma}\mu^{-(j-i)\gamma}) \leq \exp(\frac{C_2\mu^\gamma}{\mu^\gamma - 1}\frac{1}{\lambda_j^\gamma}) \\
&\leq 1 + \frac{2C_2\mu^\gamma}{\mu^\gamma - 1}\frac{1}{\lambda_j^\gamma}.
\end{aligned}$$

The desired result then follows by Theorem 7.2. ■

In the application to finite element methods, $\lambda_j \asymp h^{-2}$ with h being the finest mesh size, hence the above theorem asserts that the variable V–cycle converges if the finest mesh size is small enough.

Another more robust approach is to scale the bilinear form A_k so that (**A7.2**) is satisfied without violating (**A7.1**).

We will replace A_k by the following

$$\widetilde{A}_k(\cdot,\cdot) \stackrel{\text{def}}{=} \left(\prod_{i=1}^{k}(1+C_2\lambda_i^{-\gamma})^{-1}\right) A_k(\cdot,\cdot). \tag{7.17}$$

Then, for $v \in \mathcal{M}_{k-1}$, by (**A**7.2′)

$$\begin{aligned}
&\widetilde{A}_k(\mathcal{I}_k v, \mathcal{I}_k v) \\
&\quad= \left(\prod_{i=1}^{k}(1+C_2\lambda_i^{-\gamma})^{-1}\right) A_k(\mathcal{I}_k v, \mathcal{I}_k v). \\
&\quad= \left(\prod_{i=1}^{k}(1+C_2\lambda_i^{-\gamma})^{-1}\right) (1+C_2\lambda_k^{-\gamma})^{-1} A_{k-1}(v,v). \\
&\quad= \left(\prod_{i=1}^{k-1}(1+C_2\lambda_i^{-\gamma})^{-1}\right) A_{k-1}(v,v). \\
&\quad= \widetilde{A}_{k-1}(v,v).
\end{aligned}$$

This means that (**A7.2**) holds with $\widetilde{A}_k$ in place of A_k. It is also straightforward to see that (**A7.1**) holds with $\min(\beta,\gamma)$ in place of β. Consequently, we deduce the following

Theorem 7.3 *Under the assumption (***A7.0***), (***A7.1***) and (***A***7.2′), if the* **Algorithm S** *is modified by replacing A_k by $\widetilde{A}_k$ as given in* (7.17)*, then the results in Theorem 7.2 hold with* $\min(\beta,\gamma)$ *in place of* β*.*

Before ending this subsection, we present a result for the V–cycle that shows that the algorithm gives a preconditioner whose condition number slightly grows with j.

Theorem 7.4 (V–cycle) *Assume that (**A7.0**), (**A7.1**) and (**A**7.2′) hold and in addition* (7.16) *holds. Then, for the V-cycle of* **Algorithm S** , (7.8) *holds with* δ_k *given by* (7.10) *and* σ_k *being uniformly bounded with respec to* k. *Consequently,*

$$\kappa(B_j A_j) \lesssim 1 + j^{\frac{1}{\beta}-1} m^{-\beta}.$$

Proof. It follows from Lemma 7.3 that

$$\begin{aligned} & -A_k((I - \mathcal{I}_k \mathcal{I}_k^*) K_k^{(m)} u, K_k^{(m)} u) \\ &= A_{k-1}(\mathcal{I}_k^* K_k^{(m)} u, \mathcal{I}_k^* K_k^{(m)} u) - A_k(K_k^{(m)} u, K_k^{(m)} u) \\ &\leq C_2 {\lambda_k}^{-\gamma} A_k(K_k^{(m)} u, K_k^{(m} u) \\ &\leq C_2 {\lambda_k}^{-\gamma} A_k(u, u). \end{aligned}$$

Hence, similar to (7.14), we conclude that

$$A_k(B_k A_k v, v) \leq (1 + \sigma_k) A_k(v, v)$$

with

$$\sigma_k = \prod_{i=1}^{k} (1 + C_2 \lambda_i^{-\gamma}) \leq \prod_{i=1}^{\infty} (1 + C_2 \lambda_i^{-\gamma}) < \infty.$$

Together with Lemma 7.5, the proof is then completed. ■

7.3.3 Estimates without (A7.2) or (A7.2′): sufficiently many smoothings

A natural question to ask is that what we can say if neither (**A7.2**) nor (**A**7.2′) is satisfied. In general the multigrid process is not convergent without any further assumptions. A rather simple idea is to come back to an assumption that was always required in the earlier literature of multigrid theory, namely that the number of smoothing is sufficiently large. As a matter of fact, this is indeed enough to guarantee the convergence and we will present results for the variable V–cycle and W–cycle respectively. However it is practically troublesome to need sufficient many smoothings since it is unknown in advance how many are really enough. Fortunately, there is a more sophisticated approach to get around this problem and we will talk about it in the forthcoming subsection.

Theorem 7.5 (Variable V-cycle) *Assume only that (***A7.0***) and (***A7.1***) hold. Then for the variable V-cycle of* **Algorithm S** *, there exists a constant M such that* (7.7) *holds with*

$$\delta_k = \frac{M}{m_k^{\frac{\beta}{2}}} < 1 \tag{7.18}$$

if $m_j > M^{2/\beta}$.

Proof. It is again trivial for $k = 1$. Assume the theorem for $k-1$, it then follows from (7.5) that

$$\begin{aligned} |A_k(E_k u, u)| &\leq |A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\tilde{u}, \tilde{u})| \\ &+ \delta_{k-1} A_{k-1}(\mathcal{I}_k^* \tilde{u}, \mathcal{I}_k^* \tilde{u}). \end{aligned}$$

But

$$\begin{aligned} A_{k-1}(\mathcal{I}_k^* \tilde{u}, \mathcal{I}_k^* \tilde{u}) &= A_k(\mathcal{I}_k \mathcal{I}_k^* \tilde{u}, \tilde{u}) \\ &\leq A_k(\tilde{u}, \tilde{u}) + |A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\tilde{u}, \tilde{u})|. \end{aligned}$$

Hence

$$\begin{aligned} |A_k(E_k u, u)| &\leq (1 + \delta_{k-1})|A_k((I - \mathcal{I}_k \mathcal{I}_k^*)\tilde{u}, \tilde{u})| \\ &+ \delta_{k-1} A_k(\tilde{u}, \tilde{u}). \end{aligned}$$

Using (**A7.1**) and the estimate like (6.9), we can deduce that

$$|A_k((I-\mathcal{I}_k\mathcal{I}_k^*)\widetilde{u},\widetilde{u})| \le \frac{C}{m_k^{\frac{\beta}{2}}} A_k(u,u).$$

Consequently

$$|A_k(E_k u,u)| \le \left(\frac{(1+\delta_{k-1})C}{m_k^{\frac{\beta}{2}}} + \delta_{k-1} \right) A_k(u,u). \tag{7.19}$$

Hence if we choose M so large that

$$M \ge \frac{2C\beta_0^{\beta/2}}{\beta_0^{\beta/2}-1}.$$

then

$$\frac{(1+\delta_{k-1})C}{m_k^{\beta/2}} + \delta_{k-1} \le \delta_k.$$

This completes the proof. ■

Theorem 7.6 (*W*–cycle) *Assume only that (***A7.0***) and (***A7.1***) hold. Then for the W–cycle of* **Algorithm S** , (7.7) *holds with $\delta_k = \delta < 1$ independent of k if m is sufficiently large.*

Proof. Analogous to (7.19), we have

$$|A_k(E_k u,u)| \le \left(\frac{(1+\delta^2)C}{m^{\frac{\beta}{2}}} + \delta^2 \right) A_k(u,u). \tag{7.20}$$

Thus if

$$\delta \le \frac{1}{2},$$

and m is sufficiently large that

$$\frac{(1+\delta^2)C}{m^{\frac{\beta}{2}}} \le \frac{\delta}{2},$$

then the desired result then follows. ■

7.3.4 Estimates without (A7.2) or (A7.2′): one smoothing and optimal preconditioners

The above two theorems did not say how large m should be in order to guarantee the convergence, hence they are not very useful in practice. But we are going to show that the variable V–cycle will provide an optimal preconditioner.

Theorem 7.7 (Variable V–cycle) *Under assumptions (***A7.0***) and (***A7.1***), for variable V–cycle, there exists a constant $M > 0$, independent of k, such that* (7.8) *holds with*

$$\delta_k = \frac{M}{M + m_k^\beta}, \quad \sigma_k = \frac{M}{m_k^\beta}$$

Consequently

$$\kappa(B_j A_j) \le \frac{(M + m^\beta)^2}{m^{2\beta}}.$$

Proof. The estimate for δ_k is given by Lemma 7.5. The estimate for σ_k follows from Lemma 7.6 since

$$\tau_k - 1 \le \prod_{i=1}^{k}(1 + \frac{C_0}{m_i^\beta}) - 1 \le \frac{M}{m_k^\beta},$$

if M is sufficiently large. ■

7.3.5 Convergence with only (A7.0′) and (A7.2)

Theoretically it would be interesting to know under what conditions that the **Algorithm S** converges at all without concerning the actual convergence rate. One set of sufficient condition we found are (**A**7.0′) and (**A7.2**) which are very weak assumptions as we shall see in the next section.

Theorem 7.8 *Under the assumption (***A**7.0′*) and (***A7.2***), the* **Algorithm S** *is always convergent, namely* (7.7) *holds with $\sigma_k = 0$ and some $\delta_k \in (0, 1)$.*

Proof. By Lemma 7.4, we know that we can take $\sigma_k = 0$ in 7.7. On the other hand, (**A**7.0′) implies that $I - K_k^* K_k$ is positive, but $\mathcal{M}_k$ is finite dimensional, hence (**A7.0**) always holds if we allow $C_0 = C_0(k)$ depending on k. Similarly (**A7.1**) also holds with $C_1 = C(k)$. Therefore (**A7.0**) and (**A7.1**) both hold in this generalized sense. As a result, (7.10)-(7.12) all hold but with M depending on k, which implies $\delta_k < 1$, as desired. ■

7.4 A Convergent AMG Scheme with almost no Assumptions

In the multigrid literature, there are many papers concerning with the so-called *algebraic multigrid* (AMG) methods. This kind of method is developed to solve matrix equations using the principles of usual multigrid methods but without any background of continuous problems.

The theory of the AMG does not appear to be very satisfactory and there is no convergence theory for a reasonable class of matrices. Some AMG method may be best understood in the context of the symmetric M-matrices, but only two level convergence theory is rigorously established. c.f. Ruge and Stüben [79].

It is definitely very difficult to expect a general multilevel AMG method to converge uniformly. However we discovered that it is possible to design a general AMG method that always converges for any SPD problems. The convergence rate depends on the features of the underlined problems but should be at least as good as Richardson's iteration.

We still use the framework given in Section 7.1. As we mentioned earlier, the bilinear form A_k's can be properly scaled so that (**A7.2**) is satisfied. More precisely

$$A_k \Leftarrow (\prod_{i=k}^{j-1} s_i) A_k$$

where

$$s_i \geq \sup_{u \in \mathcal{M}_{i-1}} \frac{A_i(\mathcal{I}_i u, \mathcal{I}_i u)}{A_{i-1}(u,u)}$$

It staight forward to check that the newly defined A_k's satisfy the assumption (**A7.2**).

If we use Richardson's method as a smoother, namely $S_k = \lambda_k^{-1} I$ with λ_k being the largest eigenvalue of the scaled A_k (or an upper bound), then (**A7.0**) or (**A**7.0′) hold trivially, consequently, Theorem 7.8 is true, namely the corresponding multilevel **Algorithm S** is always convergent.

In practice, if we are only given the algebraic system on the space $\mathcal{M}$:

$$Ax = b.$$

Naturally we can take $\mathcal{M}_j \stackrel{\text{def}}{=} \mathcal{M}$ and $A_j = A$, but we need to choose $\mathcal{M}_k$ and A_k for $k < j$. Suppose the multilevel spaces $\mathcal{M}_k$ for $k < j$ are somehow defined along with some prolongation operators $\mathcal{I}_k$. There is a rather natural way to define A_k for $k < j$ as follows

$$A_{k-1}(u,v) \stackrel{\text{def}}{=} A_k(\mathcal{I}_k u, \mathcal{I}_k v), \quad \forall u, v \in \mathcal{M}_{k-1},$$

namely all A_k are inherited from A via the prolongations $\mathcal{I}_k$. In this way, (**A7.2**) is automatically satisfied. Our theory then shows that the corresponding multilevel **Algorithm S** always converges if the Richardson's method is used as a smoother.

However, the point of a multilevel algorithm is not only its convergence but, more importantly, that it converges fast enough so that the overall computing complexity is acceptable. From this point of view, the above general AMG method may not be of great practical significance. Nevertheless theoretically it indicates the robustness of the algorithm. On the other hand, it suggests that multigrid algorithms should find their applications to many more problems of wider range. As is demonstrated by the different kinds of convergence results in Section 7.2, generally speaking, the more multilevel *structure* is available, the better the algorithm can be constructed.

7.5 Comments

Abstract frameworks for multigrid algorithms for SPD problems have been proposed by many other authors. Especially for the theories that do not require the solution spaces to be nested, we refer to Douglas [40] and Mandel, McCormick and Bank [66]. One of the major improvements of our theory is relevant to (**A7.2**). In [40], the so–called *energy norm consistency* hypothesis was made as follows

$$A_k(\mathcal{I}_k v, \mathcal{I}_k v) = C_1 A_{k-1}(v, v), \quad \forall v \in \mathcal{M}_{k-1} \tag{7.21}$$

for a constant *independent* of k; convergence results were proved provided that the number of smoothing is sufficiently large (comparing Theorem 7.5). In [66], the main hypothesis is exactly (7.21) with $C_1 = 1$, namely

$$A_k(\mathcal{I}_k v, \mathcal{I}_k v) = A_{k-1}(v, v), \quad \forall v \in \mathcal{M}_{k-1}. \tag{7.22}$$

But stronger convergence results were established for any number of smoothings.

However no applications were found for trully nonnested meshes in any of the work mentioned above. In studying the multigrid algorithms for curved-boundary-domain problems, we found that (7.21) is impossible to be satisfied without violating the more crucial assumption (**A7.1**). In contrast, our theory with (**A7.2**) or even without it resolves this problem.

It seems to be a new feature that a multigrid algorithm will provide a good preconditioner even though it may not converge in the usual sense. On the other hand one can always obtain a contraction by scaling the preconditioner.

Chapter 8

Multigrid Algorithms on Finite Element Discretizations for SPD Elliptic Boundary Value Problems

In all of the preceding chapters, we have only developed preliminary materials for multigrid algorithms. The algorithms are indeed discussed in Chapter 7 but the presentation there is completely abstract. Now we are in a position to begin the discussion of some concrete applications.

This chapter is devoted to symmetric positive definite problems; thus the abstract framework developed in Chapter 7 can be used. We will only confine ourselves to the second order elliptic boundary problems with finite element discretizations which have been intensively studied in Chapter 3 and Chapter 4. Therefore we would expect to employ the materials from Chapter 3 or 4 to supply the ingredients and verify the relevant assumptions needed in the abstract framework in Chapter 7.

8.1 Preliminary

We will consider the multigrid algorithms on finite element discretization for the following second order elliptic boundary value problem:

$$\begin{aligned} \mathcal{L}U &= F \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here, Ω is a bounded domain in $\mathbb{R}^d$ and $\mathcal{L}$ is given by

$$\mathcal{L}v = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j}) + a_0 v,$$

with $\{a_{ij}\}$ uniformly positive definite and bounded on Ω and a_0 is nonnegative. Correspondingly, we have the following bilinear form:

$$A(v,w) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\ dx + a_0 vw. \tag{8.1}$$

This form is defined for all v and w in the Sobolev space $H^1(\Omega)$. Clearly, $U \in H_0^1(\Omega)$ is the solution of

$$A(U,\chi) = (F,\chi), \quad \forall \chi \in H_0^1(\Omega), \tag{8.2}$$

Throughout this chapter we again make the following elliptic regularity assumption, namely, there exists a constant $\alpha \in (0,1]$ so that

$$\|U\|_{H^{1+\alpha}(\Omega)} \leq C\|F\|_{H^{\alpha-1}(\Omega)}, \tag{8.3}$$

for the solutions U of (8.2), where C is a constant depending on the domain Ω and the coefficients defining $\mathcal{L}$.

We will use the finite element method to discretize the above problem. To do this we first need to discretize the underlying domain Ω, namely, to construct a triangulation of Ω. This procedure has been discussed in Chapter 3 in detail, hence we can assume a triangulation $\mathcal{T}_j$ is given on Ω. On this triangulation, finite element spaces can be constructed. First of all , let us consider the conforming elements. As we have done in Chapter 3, our main interests are in the continuous piecewise linear polynomial space $\mathcal{M}_j \subset H_0^1(\Omega)$. Using this space, we can formulate the finite element approximation of the problem (3.4) by

Find $u_j \in \mathcal{M}_j$ such that

$$A(u_j, v) = (f, v), \quad \forall v \in \mathcal{M}_j. \tag{8.4}$$

Another type of finite element space we will study is nonconforming element we described in Section 4.5, we denote it by $\bar{\mathcal{M}}_j$ which is the space of piecewise linear polynomials that assume the same value at the midpoint of each edge of any element and vanish at the midpoints of the edges on $\partial\Omega$. With this space, the finite element approximation of (8.2) is given by

Find $u_j \in \bar{\mathcal{M}}_j$ such that

$$A_j(u_j, v) = (f, v), \quad \forall v \in \bar{\mathcal{M}}_j. \tag{8.5}$$

Our primary purpose is to develop multigrid algorithms to solve the equation (8.4) and (8.5). More specifically, the **Algorithm S** proposed in Chapter 7 will be used. This algorithm will be studied below for different kinds of domains, triangulations, finite element spaces etc.

The **Algorithm S** consists of the following ingredients that we need to choose in different situations:

1. The multilevel spaces $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_j$.

2. Bilinear forms $A_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ for $k = 1, \cdots, j$.

3. Prolongation operators $\mathcal{I}_k : \mathcal{M}_{k-1} \mapsto \mathcal{M}_k$ for $k = 2, \cdots, j$.

In all the following applications, we will make the following assumptions, under which (**A7.2**) holds trivially with inequality:

1. Unless otherwise specified (e.g. Section 8.3), we always assume that Ω is a polygon or tetrahedron in order to construct a nested sequence of multilevel spaces.[1]

2. Unless otherwise specified (e.g. nonconforming element), all the quadratic forms $A_k(\cdot,\cdot)$ will all be the same as given by (8.1).

3. If the multilevel spaces are nested, namely $\mathcal{M}_{k-1} \subset \mathcal{M}_k$, for $k = 2, \cdots, j$, the prolongation operators $\mathcal{I}_k : \mathcal{M}_{k-1} \mapsto \mathcal{M}_k$ will be the natural inclusion operators.

The assumption (**A7.0**) on the smoothing property can be treated in a uniform way for different applications, since this assumption only involves a single space. If the smoother is the Richardson method, (**A7.0**) holds trivially. For Gauss–Seidel method, (**A7.0**) follows from 5.1, hence we have

Lemma 8.1 *For a quasiuniform mesh or nonquasiuniform mesh, (***A7.0***) always holds for Richardson method as a smoother. For quasiuniform mesh, (***A7.0***) also holds for Gauss–Seidel smoother.*

As we see, in most cases, we only need to verify (**A7.1**). To do this, we will frequently quote the results from Chapter 4 which is in fact mainly devoted to the verification of the very assumption (**A7.1**).

8.2 Nested Quasiuniform Meshes

The finite element discretization on the nested quasiuniform triangulations for the second order elliptic boundary value problems is a well studied subject in the theory of multigrid algorithms. The convergence property of various multigrid method has been studied by many others, c.f. Bramble and Pasciak [21] and

[1] For the Neumann problem, however, nested spaces may be obtained for curved boundary domains.

the references therein. Some of such known results will be derived now from our framework of Chapter 7 as a prelude of our discussion in this chapter,

To proceed, we need to assume the notations from Section 4.1. As we have done there, we construct a sequence of nested triangulations which are all quasiuniform and correspondingly we have a sequence nested spaces as follows

$$\mathcal{M}_1 \subset \mathcal{M}_2 \cdots \subset \mathcal{M}_j. \tag{8.6}$$

To define the **Algorithm S** , for each k, let $(\cdot,\cdot)_k$ be defined by (4.3), all the other ingredients needed in the algorithms are already described in the preceding section. Therefore the **Algorithm S** for our problem is defined. To apply the results in Chapter 7, we need only to verify (**A7.1**), but this is exactly Theorem 4.1 of Chapter 4. Consequently we have the following

Conclusion 8.1 *Under the assumptions described above, the results in Theorem 7.1 all hold.*

8.3 Nonnested Quasiuniform Meshes

The motivation for studying this kind of mesh is very simple: a sequence of nested meshes is not easy to set up on a domain whose boundary is curved. Hence in order to apply the multigrid idea to this kind of problem, we wnat to allow the underlying multilevel grids to be non–nested. As a result, for most of the problems (e.g. Dirichlet boundary value problem), the corresponding multilevel spaces are not imbedded as in (8.6).

However most of the results in multigrid analysis strongly rely on the imbedding property as shown in (8.6). Even though there were some theories that seemingly allow this imbedding property to be violated, their applications are limited and in particular are not appropriate to the present problem.

Now we have the abstract framework from Chapter 7, which was actually primarily motivated by this curved–boundary–domain problem. In the following we will apply the theory there to study the problem.

The ingredients needed in our development are actually prepared in Section 4.3 and 4.4 in Chapter 4. What we need to do is only to take the development in Chapter 4 and define the **Algorithm S** and verify certain assumptions required in the framework of Chapter 7.

Accordingly we have two kinds of multilevel grids, which will be discussed separately in the following.

8.3.1 Specially coupled grids

Specially coupled grids are obtained by triangulating the domain successively in a special way so that that the mesh domains more closely approximate the original domain. In particular every two successive meshes are nested except for those elements near the boundary of the domain. Detailed description is given in Section 4.3 for this kind of grid on a curved–boundary–domain. In this case, the inner product $(\cdot,\cdot)_k$ on each level is similar to the nested case, which is defined by (4.3). But prolongation operators are nodal value interpolations and they will however only perturb the value of functions in coarser spaces near the boundary of the domain because of the special structure of the multilevel grids. It turns out that assumptions (**A7.1**) and (**A**7.2′) (with $\gamma = \frac{1}{4}$) are both satisfied according to Lemma 4.5 and 4.4. Consequently, we have

Conclusion 8.2 *Under the assumptions described above, the results in Section 7.3.2 are all valid.*

8.3.2 Loosely coupled meshes

If two successive grids in a multilevel structure are not related in any intimate way, we will call these multilevel grids loosely coupled. On these grids, there is more freedom and flexibility for domains with more complex geometry. However in order to have optimal efficiency of the multigrid algorithm, the mesh size and number of nodes in each of these grids should asymptotically behave similarly to the usual nested grids. A description of loosely coupled grids is given in Section 4.4. In this case, the inner product $(\cdot,\cdot)_k$ is still defined by (4.3), but we need to make different choices of prolongations for problems of two or higher dimensions.

Two dimensional problems

In this case, prolongations are nodal value interpolations. By Lemma 4.6, assumption (**A7.1**) is satisfied, consequently

Conclusion 8.3 *Under the assumptions described above, the results in Section 7.3.3 and 7.3.4 are all valid.*

Higher dimensional problems

For higher ($d \geq 3$) dimensional problems, we have technical trouble if the prolongation is the interpolation operator. Although there are some reasons to believe that the interpolant should still be a good candidate for the prolongation in such cases, our proof in two dimensions does not carry over. As an alternative approach, we will use the L^2 quasi–projection as a prolongation. This was described in Section 4.4. By Lemma 4.5, the assumption (**A7.1**) is satisfied. Consequently,

Conclusion 8.4 *Under the assumptions described above, the results in Section 7.3.3 and 7.3.4 are all valid.*

8.4 Nested Nonquasiuniform Meshes

There are many situations where local mesh refinements are important in the finite element approximation. A typical example is the case in which the solution of the partial differential equation possesses singularities near the corner of a non–convex domain. Singularities also occur if the coefficients of the equation are discontinuous. Near a singularity, the mesh should be refined in order to maintain the accuracy. In this way, nonquasiuniform triangulations arise.

It is quite natural to try the multigrid method for the refined meshes. As a matter of fact, much attention has been paid to for this problem in the literature. Some numerical examples actually indicate the efficiency of the algorithm. However, the theoretical aspect of the algorithm gets much more complicated and it seems that very little is done in this direction. In [92] and [93], Yserentant presents some results for some systematically refined meshes, but it is not clear how to get a sequence of nested meshes that still satisfy the required conditions. Another work along this line which is sometimes mentioned in the literature is, [77], the Ph.D. thesis of von Rosendale, but unfortunately the proof appears to contain a mistake.

So far we do not know if the framework in Chapter 7 can be applied in this case for any practically useful refined meshes. The main trouble is in the verification of (**A7.1**) which is strongly related to the elliptic regularity of the underlying partial differential equation.

Our strategy in this section is somehow to skirt the elliptic regularity. Instead we will use the method presented in Chapter 6. In this way it is hard to expect to get an optimal result, but what we shall show suggests that the multigrid algorithm for refined meshes is quite reasonable and at least our theory provides a complete theoretical justification for two, three or or any fixed number of levels.

We will follow the notation and assumptions in Section 4.2. The most crucial step is perhaps to make the right choice of the inner products $(\cdot,\cdot)_k$, which can be defined by (3.7), or equivalently by

$$(u,v)_k = h_k^2 \sum_{i=1}^{J} \sum_{\tau\in\mathcal{T}_k\cap\Omega_i} h_\tau^{d-2} \sum_{x\in\mathcal{N}_k\cap\tau} u(x)v(x)$$

For $d = 2$, this coincides with what was used in [92]. But for $d > 2$, this appears to be new. This should be important for further study of higher dimensional problems.

For simplicity, we will only consider the Richardson method as the smoother.

Therefore we have all the ingredients to define the **Algorithm S** . Next we shall apply the results in Chapter 6 to establish a convergence theorem of this algorithm. To do this we need to verify the assumption (**A6.1**) and (6.14). The inequality (6.14) is trivial since the smoother of the algorithm is the simple Richardson method. By Theorem 3.9, we see that $\lambda_k \lesssim h_k^{-2}$, hence the assumption (**A6.1**) is verified by Lemma 4.2. Therefore from Lemma 7.1 and Lemma 6.3, we can conclude the following

Theorem 8.1 (Non–quasiuniform meshes) *Under assumptions in Section 4.2, the* **Algorithm S** *in Section 7.1 satisfies:*

$$0 \le A_k((I - B_k A_k)v, v) \le \delta_k A(v, v), \quad \forall v \in \mathcal{M}_k,$$

with $\delta_k = (1-\eta)\delta_{k-1} + \eta$ *and* $\delta_1 = 0$*, where* $\eta \in (0, 1)$ *is a constant independent of* k*.*

Remark 8.1 It is not hard to verify that the δ_k's given in the above theorem satisfy

$$\delta_k = 1 - (1-\eta)^{k-1}, \quad k = 1, 2, \cdots.$$

Even though δ_k tends rapidly to 1, this proves the uniform contraction of the multigrid algorithm for any locally quasiuniform meshes if the number of the levels is fixed.

8.5 Interface Problems with Large Jumps in Coefficients

This section is devoted to the following interface problem:

$$-\nabla(a\nabla)U = F \quad in\,\Omega,$$

$$U \;=\; 0 \quad on\ \partial\Omega. \tag{8.7}$$

Here $a = a(x)$ is a piecewise constant functions on Ω. More precisely Ω admits the following decomposition

$$\bar{\Omega} = \bigcup_{i=1}^{J} \bar{\Omega}_i$$

where Ω_i are mutually disjoint open polygon or tetrahedral and for $i = 1, \ldots, J$

$$a(x) = \omega_i, \qquad \forall x \in \Omega_i \tag{8.8}$$

and ω_i are positive constants.

We want to allow large jumps in the coefficient a. Evidently, the results in Section 8.2 still hold for above problem since the elliptic regularity estimate (8.3) is still valid in this case for some positive α. But one observes that the convergence rate estimate for the multigrid algorithms would depend on the jumps in a. Since these jumps may be very large, the convergence rate could be effected to a great extent so that the algorithm may be no more efficient.

In practical computations, one can use the weighted discrete L^2 as in (8.9) below to overcome this difficulty. It is observed numerically the corresponding multigrid algorithm converges independently of the jumps of a. However a theoretical justification of this phenomenon is lacking.

The whole trouble is again on the assumption (**A7.1**) if we want to apply our abstract theory in Chapter 7, since the constant C in the elliptic regularity estimate (8.3) used to prove (**A7.1**) depends on the jumps of a. Again our approach here is to use the results from Chapter 6 to study this problem.

To define the **Algorithm S** , we assume that Ω is triangulated by a nested sequence of quasiuniform meshes $\{\mathcal{T}_k, k = 1, \ldots, j\}$ as is described in Section 4.1. An obvious additional assumption we need here is that the interfaces of $\partial\Omega_i$'s are lined up with each triangulation, hence the restriction of each $\mathcal{T}_k$ on each Ω_i is also a triangulation of Ω_i itself. The discrete inner product is defined by

$$(u, v)_{k,\omega} = h_k^d \sum_{i=1}^{J} \omega_i \sum_{x \in \mathcal{N}_k \cap \bar{\Omega}_i} u(x)v(x). \tag{8.9}$$

Its induced norm, denoted by $\|\cdot\|_{k,\omega}$, is equivalent to $\|\cdot\|_{L^2_\omega(\Omega)}$ defined from (3.37).

Similarly, operators $A_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ are defined by

$$(A_k u, v)_{k,\omega} = A(u, v), \quad \forall u, v \in \mathcal{M}_k.$$

Inverse inequality implies that

$$\lambda_k \stackrel{\text{def}}{=} \rho(A_k) \lesssim h_k^{-2}.$$

Again, we will use the technique described in Chapter 6 to study this problem. To begin with, we derive from Lemma 4.1 that

Lemma 8.2

$$\|(I - I_{k-1})v\|_{k,\omega}^2 \lesssim \lambda_k^{-1} A(v, v), \quad \forall v \in \mathcal{M}_k.$$

Hence we can apply Lemma 6.3 to conclude that

Theorem 8.2 *Under assumptions described above, the* **Algorithm S** *in Section 7.1 satisfies:*

$$0 \le A_k((I - B_k A_k)v, v) \le \delta_k A(v, v), \quad \forall v \in \mathcal{M}_k,$$

$$\|E_k\|_A \le \delta_k < 1$$

with $\delta_k = 1 - (1 - \eta)^{k-1}$ *and* $\delta_1 = 0$*, where* $\eta \in (0, 1)$ *is a constant independent of* k*.*

The above theorem shows that the multigrid algorithm converges uniformly with respect to the jumps of the coefficients, provided that the number of levels is fixed. This may give a justification for some numerical experiments since in practice the number of levels are often about 3 or 4.

8.6 Interface Problems with Refined Meshes

This is a continuation of the preceding section. According to the theory of PDE, we know that the solution of the interface problems (8.7) usually possesses singularities. Hence mesh refinement are important in this case. In this way, we are confronted with two difficulties at the same time, namely the large jumps and the nonuniform grids. But fortunately our argument given above is completely local and hence still applies to this case.

The nested sequence of triangulations are given as in Section 4.2 with a constraint that the interfaces of $\partial\Omega_i$'s are lined up with each restriction $\mathcal{T}_k$. The discrete inner products on the corresponding spaces $\mathcal{M}_k$ are defined by

$$(u, v)_{k,\omega} = h_k^2 \sum_{i=1}^{j} \omega_i \sum_{\tau \in \mathcal{T}_k \cap \bar{\Omega}_i} h_\tau^{d-2} \sum_{x \in \mathcal{N}_k \cap \tau} u(x)v(x).$$

With a similar local argument, we have the analog of Lemma 4.2 as follows:

Lemma 8.3 *For any $v \in \mathcal{M}_k$,*

$$\|(I - I_{k-1})v\|_{k,\omega}^2 \lesssim h_k^2 A(v, v).$$

Therefore Lemma 6.3 is satisfied. Similar to the preceding section, we have

Theorem 8.3 (Interface problems with refined meshes) *Under assumptions described above, the* **Algorithm S** *in Section 7.1 satisfies:*

$$0 \le A_k((I - B_k A_k)v, v) \le \delta_k A(v, v), \quad \forall v \in \mathcal{M}_k,$$

with $\delta_k = 1 - (1 - \eta)^k$, where $\eta \in (0, 1)$ is a constant independent of k.

8.7 Nonconforming Finite Elements

In this section, we will design and analyze a multigrid algorithm for solving the equation (8.5). The key idea is to use the conforming elements as coarse spaces.

As usual, we are given nested triangulations as follows

$$\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_j.$$

We first assume that all the above triangulations are quasiuniform. On the finest triangulation $\mathcal{T}_j$, the Crouzeix-Raviart space $\bar{\mathcal{M}}_j$ is defined, which is a space of piecewise linear functions on $\mathcal{T}_j$ that assume the same value at two adjacent elements at the midpoint of their common edge and vanish at the midpoint of each edge on $\partial\Omega$.

To define the coarse level space, we employ the same notation $\mathcal{M}_k$ to denote the continuous piecewise linear functions on $\mathcal{T}_k$ that belongs to $H_0^1(\Omega)$. The multilevel spaces defining the **Algorithm S** will be given by

$$\tilde{\mathcal{M}}_k = \begin{cases} \bar{\mathcal{M}}_j, & \text{if } k = j; \\ \mathcal{M}_{k+1}, & \text{if } k < j, \end{cases}$$

Notice that $\tilde{\mathcal{M}}_k$ will play the role of $\mathcal{M}_k$ in the definition of **Algorithm S** .

Therefore we get a sequence of nested spaces

$$\tilde{\mathcal{M}}_1 \subset \tilde{\mathcal{M}}_2 \subset \ldots \subset \tilde{\mathcal{M}}_{j-1} \subset \tilde{\mathcal{M}}_j.$$

On $\tilde{\mathcal{M}}_j$, the bilinear form $A_j(\cdot,\cdot)$ is now defined by

$$A_j(u,v) = \sum_{\tau \in \mathcal{T}_j} A_\tau(u,v), \quad \forall u,v \in \mathcal{M}_j$$

where $A_\tau(u,v) = \int_\tau (\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv) dx$, and the bilinear form $(\cdot,\cdot)_j$ is defined by

$$(u,v)_j = h_j^2 \sum_{x \in \bar{\mathcal{N}}_j} u(x) v(x),$$

where $\bar{\mathcal{N}}_j$ = set of all midpoints of the edges in $\mathcal{T}_j$. On $\tilde{\mathcal{M}}_k$, for $k < j$, the bilinear forms $A_k(\cdot,\cdot)$ are all the same as given by (8.1) and the bilinear form $(\cdot,\cdot)_k$ is defined by

$$(u, v)_k = h_k^2 \sum_{x \in \mathcal{N}_{k+1}} u(x)v(x).$$

It follows from Lemma 4.9 and Lemma 6.3 that

Theorem 8.4 *Under the assumptions described above, the* **Algorithm S** *satisfies*

$$\|E_j\|_A \leq \delta < 1,$$

where, for variable V–cycle or W–cycle, δ is independent of j and for V–cycle, $\delta = (1-\eta)\delta_{j-1} + \eta$ for a constant $\eta \in (0,1)$ and $\delta_{j-1} \in (0,1)$ given by (7.10) with $k = j-1$.

Proof. It follows from Theorem 7.1 that

$$\|E_{j-1}\|_A \leq \delta_{j-1} < 1$$

where δ_{j-1} is given by (7.10)-(7.12) with $k = j-1$ for V–cycle, variable V–cycle and W–cycle respectively.

The desired result then follows from Lemma 7.1 and Lemma 6.3. ■

Remark 8.2 It is almost obvious that most of the results in Section 8.3 can be extended to nonconforming $\mathcal{P}_1$ elements, the details are left to interested readers.

8.8 Comments

Nonnested multigrid algorithms were also independently studied by Zhang in his Ph.D. dissertation [96]. For a special class of multilevel spaces, he proved that the W–cycle algorithm would converge if the number of smoothing is sufficiently large. Other work on some particular non–nested multigrid algorithms can be found in the paper by Brenner, c.f. [32, 33].

Chapter 9

Nonsymmetric and Indefinite Problems

In this chapter, we shall provide some new iterative convergence estimates for multigrid algorithms applied to nonsymmetric and indefinite problems.

Our attention will be mainly be paid to the analysis of the so–called symmetric scheme for which the convergence analysis for one smoothing was lacking before. However we will also take a look at the so–called non–symmetric scheme and will develop a theory that simplifies the existing work on its convergence analysis.

It turns out that our analysis for the nonsymmetric scheme is very technical. At this point we will merely present our theory for the nested meshes. A more general theory like the one in Chapter 7 and Chapter 8 needs to be further developed.

The outline of the remainder of the chapter is as follows. In Section 9.1, we describe the abstract framework to be used in this chapter. The assumptions used in our analysis and some preliminary definitions are also given there. Section 9.2 shows how this framework can be applied in the case of nonsymmetric and indefinite uniformly elliptic second order boundary value problems. In Section 9.3, we will give a simplified analysis for a symmetric scheme. The convergence estimates given in this chapter are based on three technical lemmas. In the first part of Section 9.4 we prove our multigrid theorems assuming the technical lemmas and the second part provides the proofs of the lemmas that represent the core of our analysis.

9.1 Abstract Framework and Assumptions

In this section, we first give an abstract framework for our nonsymmetric multigrid application. This abstract presentation more clearly identifies the relevant hypotheses used in the iterative convergence analysis to be developed. We then list the assumptions required for the multigrid analysis presented in later sections. To keep the presentation from becoming too abstract, we show how a model application to a second order problem fits into this framework in the next section.

We start with a Hilbert scale (cf. [57]) of spaces $\{H^\gamma\}$ for $\gamma \in [0,2]$. The norm on H^γ will be denoted by $\|\cdot\|_{H^\gamma}$. We assume that $H^s \subset H^t$ whenever $t < s$. The largest space (i.e. $\gamma = 0$) will be denoted H with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)$. The space H^γ is assumed to be compactly contained in H^δ whenever $\gamma > \delta$. Let $\mathcal{M}$ be a closed subspace of H^1. The spaces H^s for $-1 \leq s < 0$ are defined by duality and with norm given by

$$\|v\|_{H^s} \equiv \sup_{\phi \in \mathcal{M}} \frac{(v,\phi)}{\|\phi\|_{H^{-s}}}.$$

Assume that we are given a nested sequence of 'approximation' subspaces

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_j \subset \mathcal{M}.$$

In addition, let $\hat{A}(\cdot,\cdot)$ be a positive definite symmetric bilinear form on $\mathcal{M} \times \mathcal{M}$ satisfying

$$\hat{A}(v,v) \asymp \|v\|^2_{H^1} \quad \forall v \in \mathcal{M} \tag{9.1}$$

and $D(\cdot,\cdot)$ be a bilinear form on $\mathcal{M} \times \mathcal{M}$. We shall be interested in approximating the solution of

$$A(u,\phi) \equiv \hat{A}(u,\phi) + D(u,\phi) = (f,\phi) \quad \forall \phi \in \mathcal{M}, \tag{9.2}$$

for a given function $f \in H$. We shall assume that (9.2) is uniquely solvable for any $f \in H$.

We will be interested in applying multigrid procedures to develop a rapidly converging iterative algorithm for the solution of the Galerkin approximation of (9.2) in the subspace $\mathcal{M}_j$. Specifically, we seek the function $u \in \mathcal{M}_j$ which satisfies

$$A(u, \chi) = (f, \chi) \quad \forall \chi \in \mathcal{M}_j. \tag{9.3}$$

We next list the assumptions required for our multigrid analysis.

(**A9.1**) The first assumption involves elliptic regularity for the forms $A(\cdot, \cdot)$ and $\hat{A}(\cdot, \cdot)$. We assume that solutions u of (9.2) and the corresponding equation

$$\hat{A}(u, \theta) = (f, \theta) \quad \forall \theta \in \mathcal{M}$$

satisfy

$$\|u\|_{H^{1+\alpha}} \lesssim \|f\|_{H^{\alpha-1}} \tag{9.4}$$

for some $\alpha \in (3/4, 1]$ independent of f.

(**A9.2**) We assume that D satisfies

$$|D(v, w)| \lesssim \|v\|_{H^1} \|w\|_H \quad \forall v, w \in \mathcal{M}. \tag{9.5}$$

It is an immediate consequence of (9.5) that the operator $D : \mathcal{M} \mapsto H$ defined by

$$(Dv, \theta) = D(v, \theta) \quad \forall \theta \in H$$

is well defined and satisfies

$$\|Dv\|_H \lesssim \|v\|_{H^1}. \tag{9.6}$$

We further assume that D maps, for $\alpha \in (0, 1]$, $H^{1+\alpha}$ into H^α, i.e.

$$\|Dv\|_{H^\alpha} \lesssim \|v\|_{H^{1+\alpha}}. \tag{9.7}$$

Let $D^* : H \mapsto H^{-1}$ be defined by

$$(D^* w, \phi) = (w, D\phi) .$$

We assume that D^* is a bounded operator from H^1 into $H^{-1/2-\epsilon}$ for any positive ϵ.

(**A9.3**) We require approximation properties for the subspaces $\{\mathcal{M}_k\}$. These are given in terms of a parameter h_k which satisfies

$$h_k \asymp \eta^k, \quad \text{for a constant } \eta \in (0,1).$$

We assume that for v in H^s and $s \in [1, 1+\alpha]$, there exists $\chi \in \mathcal{M}_k$ such that

$$\|v - \chi\|_H + h_k \|v - \chi\|_{H^1} \lesssim h_k^s \|v\|_{H^s}.$$

(**A9.4**) We require that the inverse inequality,

$$\|w\|_{H^\beta} \lesssim h_k^{\gamma-\beta} \|w\|_{H^\gamma} \quad \forall w \in \mathcal{M}_k$$

hold for all $\beta > \gamma$ with $\beta, \gamma \in [0, 1+\alpha]$.

(**A9.5**) We require first that the discrete inner product $(\cdot,\cdot)_k$ be equivalent to $(\cdot,\cdot)$ on $\mathcal{M}_k$, i.e.

$$\|\chi\|_k \asymp \|\chi\|_H. \tag{9.8}$$

In addition, we assume that the discrete inner products accurately approximate the inner product on H in the sense that

$$|(\psi, \chi) - (\psi, \chi)_k| \lesssim h_k \|\psi\|_{H^1} \, \|\chi\|_k \quad \forall \psi, \chi \in \mathcal{M}_k. \tag{9.9}$$

We next introduce some discrete operators which play a fundamental role both in the analysis and the algorithms to be considered in this chapter:

The operators $A_k, \hat{A}_k, D_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ are defined for $w, \theta \in \mathcal{M}_k$, by

$$\begin{aligned} (A_k w, \theta)_k &= A(w, \theta), \\ \left(\hat{A}_k w, \theta\right)_k &= \hat{A}\,(w, \theta)\,, \\ (D_k w, \theta)_k &= D\,(w, \theta)\,. \end{aligned}$$

The operators $P_k, \hat{P}_k, \Pi_k : \mathcal{M} \mapsto \mathcal{M}_k$ are defined for $w \in \mathcal{M},\ \theta \in \mathcal{M}_k$, by

$$\begin{aligned} A\left(P_k w, \theta\right) &= A\left(w, \theta\right), \\ \hat{A}\left(\hat{P}_k w, \theta\right) &= \hat{A}\left(w, \theta\right), \\ (\Pi_k w, \theta)_k &= (w, \theta). \end{aligned}$$

The operator $I_k^{k-1} : \mathcal{M}_k \mapsto \mathcal{M}_{k-1}$ is defined for $w_k \in \mathcal{M}_k$ and $\theta_{k-1} \in \mathcal{M}_{k-1}$, by

$$(I_k^{k-1} w_k, \theta_{k-1})_{k-1} = (w_k, \theta_{k-1})_k.$$

All of the above operators except possibly P_k are clearly well defined. we shall assume, however, that h_k is less than some positive constant ν with ν chosen small enough so that the above assumptions imply that P_k is uniquely defined (cf. [80]). This also implies that A_k is invertible.

We note that (9.3) is equivalent to

$$A_k u = \Pi_k f.$$

We define two scales of norms on $\mathcal{M}_k$ which we shall use in our analysis. The operator $\hat{A}_k$ is symmetric and positive definite on $\mathcal{M}_k$ in the $(\cdot, \cdot)_k$ inner product. We define the scale of norms $\{\|\cdot\|_{k,s}\}$ for any real s by

$$\|w\|_{k,s} = \left\|\hat{A}_k^{s/2} w\right\|_k \quad \forall w \in \mathcal{M}_k.$$

Similarly, the operator ${A_k}^t A_k$ is also symmetric and positive definite on $\mathcal{M}_k$ (here, t denotes the adjoint with respect to $(\cdot, \cdot)_k$). We define the scale of norms $\{\interleave \cdot \interleave_{k,s}\}$ for any real s by

$$\interleave w \interleave_{k,s} = \left(({A_k}^t A_k)^{s/2} w, w\right)_k^{1/2} \quad \forall w \in \mathcal{M}_k.$$

Let $L_k = ({A_k}^t A_k)^{1/2}$ then clearly

$$\interleave w \interleave_{k,s} = \left\|L_k^{s/2} w\right\|_k \quad \forall w \in \mathcal{M}_k.$$

We will often consider the norms of operators from a space into itself. If $T : S \mapsto S$ is an operator on a generic space S with norm $\|\cdot\|$ then the norm of T will be denoted by $\|T\|$ and is given by

$$\|T\| = \sup_{\phi \in S} \frac{\|T\phi\|}{\|\phi\|}.$$

9.2 An Application to the Second Order Problem

We consider a model second order problem in this section and show that the hypotheses of Section 2 are satisfied. This application involves a finite element approximation of a nonsymmetric and indefinite elliptic problem in N dimensional Euclidean space.

Let Ω be a domain in R^N. The spaces $H^s = H^s(\Omega)$ will be the Sobolev spaces of order s on Ω [58, 73]. We shall be interested in approximating the solution of the problem

$$\mathcal{L}U = F, \quad \text{in } \Omega, \tag{9.10}$$

$$U = 0, \quad \text{on } \partial\Omega, \tag{9.11}$$

$$\tag{9.12}$$

where

$$\mathcal{L}U = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial U}{\partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial U}{\partial x_i} + c(x)U.$$

We assume that the matrix $\{a_{ij}(x)\}$ is symmetric and uniformly positive definite.

Under appropriate smoothness assumptions for the domain Ω and coefficients defining $\mathcal{L}$, it is possible to prove that the solutions of (9.10)–(9.11) satisfy estimates of the form (9.4) [45, 55]. For two dimensional polygonal domains, with coefficients in $C^1(\Omega)$, (9.4) holds for $\alpha > 3/4$, if all interior angles of the polygon are bounded by $4\pi/3$. For more general applications, we implicitly assume the appropriate hypotheses so that (9.4) holds for $\alpha > 3/4$.

The space $\mathcal{M}$ is a subset of $H^1(\Omega)$ satisfying appropriate boundary conditions. In the case of boundary condition (9.11), $\mathcal{M}$ is the completion of $C_0^\infty(\Omega)$ in the $H^1(\Omega)$ norm.

A weak formulation of (9.10)–(9.11) is: Find $u \in \mathcal{M}$ such that

$$A(U,\chi) = (F,\chi) \quad \forall \chi \in \mathcal{M} \tag{9.13}$$

where $(\cdot,\cdot)$ is the usual $L^2(\Omega)$ inner product and

$$\begin{aligned} A(u,v) \quad &= \quad \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \sum_{i=1}^{N} \int_\Omega b_i \frac{\partial u}{\partial x_i} v\, dx \\ &+ \quad \int_\Omega cuv\, dx \end{aligned}$$

Note that, in general, $A(\cdot,\cdot)$ is nonsymmetric and indefinite. We assume that (9.13) has a unique solution.

We define $\hat{A}\,(\cdot,\cdot)$ by

$$\hat{A}\,(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \int_\Omega uv\, dx.$$

Then, obviously

$$D\,(u,v) = \sum_{i=1}^{N} \int_\Omega b_i \frac{\partial u}{\partial x_i} v\, dx + \int_\Omega (c-1)uv\, dx.$$

We next check (**A9.2**). Inequality (9.5) follows immediately from the Schwartz inequality. The operator D is given by

$$Du = \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} + (c-1)u$$

and hence (9.7) clearly holds. Finally, we note that for $w \in H^1(\Omega)$ and $\phi \in \mathcal{M}$,

$$(D^*w, \phi) = (w, D\phi) = -(Dw, \phi) + \sum_{i=1}^{N}\{\int_{\partial\Omega} b_i n_i w\phi\, ds - (\frac{\partial b_i}{\partial x_i} w, \phi)\}, \qquad (9.14)$$

where n_i is the component of the outward normal in the i'th direction. We assume that b_i is in $C^1(\Omega)$ and that c is in $L^\infty(\Omega)$. The boundary term in (9.14) vanishes in the case of boundary conditions (9.11) and hence $D^* : H^1(\Omega) \mapsto L^2(\partial\Omega)$ in this case.

We next consider the finite element approximation subspaces. For simplicity, we shall only describe a piecewise linear application in two dimensions. The application to higher dimensional problems and more general approximation subspaces is straightforward. The nested sequence of triangulations and the corresponding finite element spaces are constructed as in Section 4.1. The standard techniques in the theory of finite elements (c.f. Section 3.4 and 3.8) imply that (**A9.3**) and (**A9.4**) hold.

We finally define the discrete inner products. Let $x_{ij}^k,\ j = 1, 2, 3$ denote the vertices of the i'th triangle of the k'th grid. Define

$$(\phi, \chi)_k = 1/3 \sum_i \left|\tau_i^k\right| \sum_{j=1}^{3} \phi(x_{ij}^k)\chi(x_{ij}^k). \qquad (9.15)$$

Here $\left|\tau_i^k\right|$ denotes the area of the triangle τ_i^k. It is not difficult to show that (**A9.5**) holds for this inner product. Note that (9.15) can be rewritten

$$(\phi, \chi)_k = \sum_i \omega_i^k \phi(y_i^k)\chi(y_i^k) \qquad (9.16)$$

where $\{y_i^k\}$ are the nodes of the k'th grid and ω_i^k is an appropriate weight function.

We will define the multigrid algorithms in this section and develop certain recurrence relations which will be used in the iterative convergence analysis given later in the chapter. The multigrid algorithm defines a linear operator B_k on $\mathcal{M}_k$ which is an approximate inverse for A_k.

We define the operator $B_k : \mathcal{M}_k \mapsto \mathcal{M}_k$ by induction on k in the following algorithm.

A Multigrid Algorithm

Set $B_1 = A_1^{-1}$. Assume that $B_{k-1} : \mathcal{M}_{k-1} \mapsto \mathcal{M}_{k-1}$ has been defined and define $B_k g$ for $g \in \mathcal{M}_k$ and $k = 1, \dots j$ as follows:

1. *Set $x^0 = 0$ and $q^0 = 0$.*

2. *For $l = 1, \dots, m_k$, define, for the nonsymmetric scheme,*

$$x^l = x^{l-1} + \mu_k^{-1}(g - A_k x^{l-1}); \tag{9.17}$$

and for the symmetric scheme,

$$x^l = x^{l-1} + \mu_k^{-2} {A_k}^t (g - A_k x^{l-1}), \tag{9.18}$$

where λ_k is the spectral radius of A_k and μ_k is the largest eigenvalue of $L_k = ({A_k}^t A_k)^{1/2}$.

3. *Define $B_k g = x^{m_k} + q^p$ where q^i, for $i = 1, 2, \dots, p$, is defined by*

$$q^i = q^{i-1} + B_{k-1}[I_{k-1}(g - A_k x^{m_k}) - A_{k-1} q^{i-1}]. \tag{9.19}$$

Remark 9.1 We have used μ_k^2 in (9.18) for convenience. In a practical implementation, any reasonable bound for the largest eigenvalue of the system ${A_k}^t A_k$ can be used.

As in Chapter 7, the following relation provides a fundamental identity for the analysis of the multigrid algorithm:

$$I - B_k A_k = [(I - P_{k-1}) + (I - B_{k-1} A_{k-1})^p P_{k-1}] {K_k}^{m_k}, \tag{9.20}$$

where for nonsymmetric scheme $K_k = I - \lambda_k^{-1} A_k$ and for symmetric scheme $K_k = I - \mu_k^{-2} A_k^t A_k$.

The goal of this chapter is to prove inequalities of the form

$$\|I - B_k A_k\|_{k,1}^2 \leq \delta_k. \tag{9.21}$$

Such inequalities immediately imply that the linear iteration

$$u^{n+1} = u^n + B_k(F - A_k u^n)$$

converges to the solution u of

$$A_k u = F$$

with a rate of $\sqrt{\delta_k}$ per step in the norm $\|\cdot\|_{k,1}$. Equality (9.20) gives a way of relating the reduction δ_k to that of the $(k-1)$-grid and hence provides a key ingredient for a mathematical induction argument.

9.3 Convergence Analysis for a Nonsymmetric Scheme

In this section, we give an alternative proof of a result by Mandel [64]. The proof below by the author seems to be somewhat simpler. Note that, in the analysis of nonsymmetric scheme, the assumption (9.9) is not necessary.

We begin with a technical lemma.

Lemma 9.1 *Let $K_k = I - \lambda_k^{-1} A_k$, $\hat{K}_k = I - {\lambda_k}^{-1} \hat{A}_k$ and $V_k = {K_k}^m - \hat{K}_k^m$, then*

$$\|V_k \hat{A}_k^{-1/2}\|_k \lesssim \sigma_m h_k^2,$$

where $\sigma_m = \frac{(1+c_0 h_1)^m - 1}{c_0 h_1}$.

Proof. Note that

$$\begin{aligned} \|K_k\|_k &\leq 2, \\ \|\lambda_k{}^{-1} D_k\|_k &\leq c_0 h_1, \\ \hat{A}_k \hat{K}_k &= \hat{K}_k \hat{A}_k. \end{aligned}$$

Hence

$$\begin{aligned} \|V_k \hat{A}_k^{-1/2}\|_k &= \|((\hat{K}_k + \lambda_k{}^{-1} D_k)^m - \hat{K}_k^m)\hat{A}_k^{-1/2}\|_k \\ &\leq \sum_{i=1}^{m} C_m^i \|\lambda_k{}^{-1} D_k\|_k{}^{i-1} \|\lambda_k{}^{-1} D_k\|_k \\ &\leq C h_k^2 \sum_{i=1}^{m} C_m^i (c_0 h_1)^{i-1} = C\sigma_m h_k^2. \end{aligned}$$

■

Theorem 9.1 *Assume $\alpha = 1$ in the assumption (**A9.1**). For the V–cycle non-symmetic scheme with $m_k \equiv m$, there exist constants C_0, C_1, C_2 such that*

$$\|I - B_k A_k\|_{k,1} \leq \delta$$

where

$$\delta = \frac{C_1^2}{2m + C_1^2} + (2^\gamma - 1)^{-1} h_1^\gamma + C_2 m h_1 \sigma_m (h_1 \sigma_m + 1).$$

Therefore for any fixed $m \geq 1$,

$$\|I - B_k A_k\|_{k,1} \leq \delta < 1$$

when h_1 is sufficiently small.

Proof. We will use an induction argument to show that

$$\|I - B_k A_k\|_{k,1}^2 \leq \delta_k^2 \tag{9.22}$$

with

$$\delta_k^2 = \frac{C_1^2}{2m + C_1^2} + \sum_{i=1}^{k} (h_i^\gamma + C_2 m h_i \sigma_m (h_i \sigma_m + 1)) \tag{9.23}$$

which implies the theorem since a trivial computation shows that

$$\delta_k \leq \delta.$$

As usual (9.22) is trivial for $k = 1$. Assuming it holds for $k-1$, we can then deduce that

$$\|I - B_k A_k\|_{k,1}^2 \tag{9.24}$$

$$\leq \quad C(1 - \delta_{k-1})(\lambda_k^{-1} \|\hat{A}_k \widetilde{u}\|_k^{\,2}) + (\delta_{k-1} + h^\gamma) \|\hat{A}_k^{1/2} \widetilde{u}\|_k^{\,2}. \tag{9.25}$$

But

$$\begin{aligned} \|\hat{A}_k \widetilde{u}\|_k^{\,2} &= \|\hat{A}_k (\hat{K}_k^m + V_k) u\|_k^{\,2} \\ &= \|\hat{A}_k \hat{K}_k^m u\|_k^{\,2} + \|\hat{A}_k V_k u\|_k^{\,2} + 2(\hat{A}_k \hat{K}_k^m u, \hat{A}_k V_k u)_k. \end{aligned}$$

It follows from Lemma 9.1 that

$$\|\hat{A}_k V_k u\|_k \lesssim h_k^{-2} \|V_k u\|_k \lesssim \sigma_m \|\hat{A}_k^{1/2} \widetilde{u}\|_k$$

and it is clear that

$$\|\hat{A}_k \hat{K}_k^m u\|_k \leq \|\hat{A}_k u\|_k \lesssim h_k^{-1} \|\hat{A}_k^{1/2} u\|_k.$$

Combining the above three inequalities yields

$$\|\hat{A}_k \widetilde{u}\|_k^{\,2} \leq \|\hat{A}_k \hat{K}_k^m u\|_k^{\,2} + C\sigma_m(\sigma_m + h_k^{-1}) \|\hat{A}_k^{1/2} u\|_k. \tag{9.26}$$

Analogously

$$\begin{aligned} \|\hat{A}_k^{1/2} \widetilde{u}\|_k &= \|\hat{A}_k^{1/2} \hat{K}_k^m u\|_k^{\,2} + \|\hat{A}_k^{1/2} V_k u\|_k^{\,2} + 2(\hat{A}_k \hat{K}_k^m u, V_k u)_k \\ &\leq \|\hat{A}_k^{1/2} \hat{K}_k^m u\|_k^{\,2} + C(h_k \sigma_m)^2 \|\hat{A}_k^{1/2}\|_k^{\,2}. \end{aligned}$$

Applying (9.26) and the above inequality to (9.24), we deduce that

$$
\begin{aligned}
& \|I - B_k A_k\|_{k,1}^2 \\
\leq\ & C_1^2(1-\delta_{k-1})\{h_k^2\|\hat{A}_k\hat{K}_k^m u\|_k^{\ 2} + Ch_k\sigma_m(h_k\sigma_m+1)\|\hat{A}_k^{1/2}u\|_k\} \\
+\ & (\delta_{k-1}+h^\gamma)\{\|\hat{A}_k^{1/2}\hat{K}_k^m u\|_k^{\ 2} + C(h_k\sigma_m)^2\|\hat{A}_k^{1/2}u\|_k^{\ 2}\} \\
\leq\ & C_1^2(1-\delta_{k-1})\{\frac{1}{2m}(\hat{A}_k(I-K_k^{\ 2m})u,u) + Ch_k\sigma_m(h_k\sigma_m+1)\|\hat{A}_k^{1/2}u\|_k\} \\
+\ & (\delta_{k-1}+h^\gamma)\left(\|\hat{A}_k^{1/2}\hat{K}_k^m u\|_k^{\ 2} + C(h_k\sigma_m)^2\|\hat{A}_k^{1/2}u\|_k^{\ 2}\right).
\end{aligned}
$$

Consequently, if

$$
C_1^2\frac{1-\delta_{k-1}}{2m} \leq \delta_{k-1} \tag{9.27}
$$

then

$$
\begin{aligned}
& \|I - B_k A_k\|_{k,1}^2 \\
\leq\ & (\delta_{k-1}+h_k^\gamma + C_2 m h_k\sigma_m(h_k\sigma_m+1))\|\hat{A}_k^{1/2}u\|_k^{\ 2},
\end{aligned}
$$

Thus it suffices to have

$$
\delta_k \geq \delta_{k-1} + h_k^\gamma + C_2 m h_k\sigma_m(h_k\sigma_m+1). \tag{9.28}
$$

It is evident that (9.27) and (9.28) will be satisfied if we choose δ_k as in (9.23). ■

Remark 9.2 Results similar to Theorem 9.1 also hold for W–cycle algorithms with a similar proof. In general it is unlikely the variable V–cycle would also converge since the frequences corresponding to the negative eigenvalues of A_k will be amplified if the number of smoothings is too large. In contrast, the symmetric scheme converges for the variable V–cycle (c.f. Theorem 9.2).

9.4 Convergence Analysis for the Symmetric Scheme

The symmetric scheme will be carefully analyzed in this section. The technique used is different from that in the preceding section for nonsymmetric scheme.

We first give results for the variable V–cycle. Next, we consider the V–cycle with constant $m(k) = m$. Finally, we consider the W–cycle algorithm. The proofs of these theorems depend on three lemmas. These lemmas are central to the analysis and will be proved in the next subsection. In this section, we prove our multigrid theorems assuming the lemmas.

9.4.1 The convergence theorems and their proofs

We start by stating the lemmas. The first lemma gives a so called 'regularity and approximation' estimate for the projection operator P_k.

Lemma 9.2 *Assume that h_1 is sufficiently small. Then*

$$\|(I - P_{k-1})v\|_1^2 \lesssim (\mu_k^{-1} \|L_k v\|_k^2)^\alpha \, (L_k v, v)_k^{1-\alpha} \quad \forall v \in \mathcal{M}_k.$$

The next two lemmas represent an essential part of the analysis of this paper. Their proof uses the Dunford-Taylor integral formula for operators and is given in the next section.

Lemma 9.3 *Assume that h_1 is sufficiently small. Then, for all $v \in \mathcal{M}_k, \chi \in \mathcal{M}_{k-1}$,*

$$(L_k(I - P_{k-1})v, \chi)_k \lesssim h_k^{\alpha-1/2-\epsilon} |||(I - P_{k-1})v|||_{k,1} |||\chi|||_{k,1} \tag{9.29}$$

holds for any positive ϵ.

Lemma 9.4 *Assume that h_1 is sufficiently small. Then, for all $\chi \in \mathcal{M}_{k-1}$,*

$$|\left| |||\chi|||_{k,1}^2 - |||\chi|||_{k-1,1}^2 \right|| \lesssim h_k^{4\alpha-3}(1 + |\log h_k|) |||\chi|||_{k,1}^2.$$

We can now state and prove the convergence theorem for the variable V–cycle algorithm with (9.18).

Theorem 9.2 *Let $p = 1$ and assume that m_k satisfies*

$$\gamma_0 m_k \leq m_{k-1} \leq \gamma_1 m_k, \tag{9.30}$$

where γ_0 and γ_1 are constants greater than one and independent of k, for $k = 2, \ldots, j$. Let γ be positive and less than $\min(\alpha - 1/2, 4\alpha - 3)$. Then there exist positive constants M and ν not depending on k such that when $h_1 \leq \nu$, (9.21) holds with

$$\delta_k = \frac{M}{M + m_k^{\alpha/2}} \tag{9.31}$$

for $k = 1, \ldots, j$.

Proof. We will prove the theorem by induction. Clearly, (9.21) holds for $k = 1$ with δ_k given by (9.31). Let $k \in \{1, \ldots, j\}$ and assume that (9.21) holds for $k - 1$ with δ_{k-1} given by (9.31). It follows from the recurrence relation (9.20) that

$$\begin{aligned} |||(I - B_k A_k)v|||_{k,1}^2 &= |||(I - P_{k-1})\tilde{v}|||_{k,1}^2 + |||(I - B_{k-1}A_{k-1})P_{k-1}\tilde{v}|||_{k,1}^2 \\ &+ 2(L_k(I - P_{k-1})\tilde{v}, (I - B_{k-1}A_{k-1})P_{k-1}\tilde{v})_k. \end{aligned}$$

where $\tilde{v} = {K_k}^{m_k} v$. Applying Lemma 9.3 gives

$$|||(I - B_k A_k)v|||_{k,1}^2 \leq (1 + Ch_k^{\alpha - 1/2 - \epsilon}) \left(|||(I - P_{k-1})\tilde{v}|||_{k,1}^2 + |||(I - B_{k-1}A_{k-1})P_{k-1}\tilde{v}|||_{k,1}^2 \right).$$

Using Lemma 9.4 and the induction hypothesis, we deduce that

$$\begin{aligned} |||(I - B_{k-1}A_{k-1})P_{k-1}\tilde{v}|||_{k,1}^2 &\leq (1 + Ch_k^{\gamma}) |||(I - B_{k-1}A_{k-1})P_{k-1}\tilde{v}|||_{k-1,1}^2 \\ &\leq \delta_{k-1}(1 + Ch_k^{\gamma}) |||P_{k-1}\tilde{v}|||_{k-1,1}^2 \\ &\leq \delta_{k-1}(1 + Ch_k^{\gamma}) |||P_{k-1}\tilde{v}|||_{k,1}^2, \end{aligned}$$

holds for any fixed γ less than $4\alpha - 3$. It follows from Lemma 9.3 that

$$\begin{aligned} |||\tilde{v}|||_{k,1}^2 &= |||P_{k-1}\tilde{v}|||_{k,1}^2 + |||(I - P_{k-1})\tilde{v}|||_{k,1}^2 + 2\left(L_k(I - P_{k-1})\tilde{v}, P_{k-1}\tilde{v}\right)_k \\ &\geq (1 - Ch_k^{\alpha - 1/2 - \epsilon})(|||P_{k-1}\tilde{v}|||_{k,1}^2 + ||| \left(I - P_{k-1})\tilde{v}|||_{k,1}^2\right) \end{aligned}$$

and thus for ν sufficiently small

$$\|P_{k-1}\tilde{v}\|_{k,1}^2 \leq (1 + Ch_k^{\alpha-1/2-\epsilon})\|\tilde{v}\|_{k,1}^2 - \|(I - P_{k-1})\tilde{v}\|_{k,1}^2.$$

Requiring, in addition, that $\gamma < \alpha - 1/2$ and combining the above inequalities gives

$$\|(I - B_k A_k)v\|_{k,1}^2 \leq (1 + Ch_1^\gamma)\left\{(1 - \delta_{k-1})\|(I - P_{k-1})\tilde{v}\|_{k,1}^2 + \delta_{k-1}\|\tilde{v}\|_{k,1}^2\right\}.$$

By Lemma 9.2, the Schwarz inequality, and a generalized arithmetic geometric mean inequality,

$$\begin{aligned} \|(I - P_{k-1})\tilde{v}\|_{k,1}^2 &\lesssim \left(\mu_k^{-1}\left(L_k^2\tilde{v}, \tilde{v}\right)_k\right)^\alpha (L_k\tilde{v}, \tilde{v})_k^{1-\alpha} \\ &\lesssim \left(\mu_k^{-2}\left(L_k^3\tilde{v}, \tilde{v}\right)_k\right)^{\alpha/2} (L_k\tilde{v}, \tilde{v})_k^{1-\alpha/2} \\ &= (L_k(I - K_k){K_k}^{m_k}v, {K_k}^{m_k}v)_k^{\alpha/2} (L_k\tilde{v}, \tilde{v})_k^{1-\alpha/2} \end{aligned}$$

Applying the Lemma 6.1 with $\epsilon = Ch_1^\gamma$ then completes the proof of the theorem. ■

We next prove a theorem for the standard V–cycle algorithm.

Theorem 9.3 *Consider the V–cycle algorithm ($p = 1$) with $m_k = m$ for all k. Let γ be positive and less than* $\min(\alpha - 1/2, 4\alpha - 3)$. *Then there exist positive constants M, c, and ν not depending on k such that when $h_1 \leq \min(\nu, c(j-1)^{-2/(\alpha\gamma)})$,* (9.21) *holds with*

$$\delta_k = \frac{Mk^{(2-\alpha)/\alpha}}{Mk^{(2-\alpha)/\alpha} + m^{\alpha/2}} \tag{9.32}$$

for $k = 1, \ldots, j$.

Remark 9.3 The theorem suggests that the V–cycle may be less robust than the variable V–cycle. Note that the convergence estimate for the V–cycle algorithm deteriorates as k becomes larger even in the case $\alpha = 1$. Furthermore, the theorem suggests that for stability, the coarsest grid must become finer as the number of grid levels increase.

Using the same proof as above we can get the theorem for the W–cycle algorithm as follows.

Theorem 9.4 *Consider the W–cycle algorithm ($p = 2$) with $m_k \equiv m$ for all k. Let γ be positive and less than* $\min(\alpha - 1/2, 4\alpha - 3)$. *Then there exist positive constants M and ν such that when $h_1 \leq \nu$,* (9.21) *holds with*

$$\delta_k \equiv \delta = (1 + m/M)^{-\alpha/2} \tag{9.33}$$

for $k = 1, \ldots, j$.

9.4.2 Proofs of the Lemmas

This section will provide the proofs of Lemmas 9.2-9.4. Before proceeding, let us state two propositions and two preliminary propositions.

Proposition 9.1 *For all $v \in H^1$,*

$$\|v\|_{H^1}^2 \lesssim A(v, v) + \|v\|_H^2.$$

Proposition 9.2 *For $v \in H^{1+\alpha}$ and $0 \leq \delta \leq \alpha$,*

$$\|(I - \hat{P}_k)v\|_{H^{1-\delta}} \lesssim h_k^\delta \|(I - \hat{P}_k)v\|_{H^1}. \tag{9.34}$$

If h_1 is sufficiently small, then P_k is well defined and

$$\|(I - P_k)v\|_{H^1} \lesssim \inf_{\chi \in \mathcal{M}_k} \|v - \chi\|_{H^1}, \tag{9.35}$$

for all $v \in \mathcal{M}$.

Proposition 9.1 follows immediately from (9.5). (9.34) follows from a standard duality argument (c.f. Proposition 3.8) and (9.35) can be proved by using the techniques given in [80].

We next introduce the preliminary lemmas. The first lemma was essentially proved in [7].

Lemma 9.5 *Let* $0 \le s \le 1$. *Then*

$$\|\chi\|_{H^s} \asymp |\!|\!|\chi|\!|\!|_{k,s} \asymp \|\chi\|_{k,s} \quad \forall \chi \in \mathcal{M}_k.$$

In addition,

$$|\!|\!|\chi|\!|\!|_{k,2} \asymp \|\chi\|_{k,2} \quad \forall \chi \in \mathcal{M}_k.$$

Lemma 9.6

$$\|(I - \Pi_k)v\|_H \lesssim h_k \|v\|_{H^1}, \tag{9.36}$$

$$\|\Pi_k v\|_{H^s} \lesssim \|v\|_{H^s} \text{ for } 0 \le s \le 1. \tag{9.37}$$

Proof. The proof is identical to that of Lemma 3.6. ■

We can now prove Lemma 9.2.

Proof of Lemma 9.2

Following the argument in Section 4.1.1, we can easily show (using our assumptions and definitions) that

$$\|(I - \hat{P}_{k-1})v\|_1^2 \lesssim (h_k^2 \left\|\hat{A}_k v\right\|_k^2)^\alpha \hat{A}(v,v)^{1-\alpha} \quad \forall v \in \mathcal{M}_k.$$

We note that (**A9.4**) and Lemma 9.5 imply that $h_k^2 \lesssim \mu_k^{-1}$. The lemma now follows from (9.35) and Lemma 9.5.

The proofs of Lemma's 9.3 and 9.4 require some technical perturbation estimates. We consider the term on the left hand side of (9.29). Let $G_k = L_k - A_k$, then since

$$(A_k(I - P_{k-1})v, \chi)_k = 0,$$

we have

$$(L_k(I - P_{k-1})v, \chi)_k = (G_k(I - P_{k-1})v, \chi)_k = \Big((I - P_{k-1})v, G_k^t \chi\Big)_k$$
$$\leq \|(I - P_{k-1})v\|_k \left\|G_k^t \chi\right\|_k . \qquad (9.38)$$

Thus, we must estimate $G_k^t = L_k - \hat{A}_k - D_k^t$.

In light of (9.38), we see that it would be useful to estimate the difference $L_k - \hat{A}_k$. Note that L_k is defined as the positive square root of the discrete operator $L_k^2 \equiv {A_k}^t A_k$. An alternative expression for L_k is given by the Dunford-Taylor integral representation (cf. [52]);

$$L_k = (2\pi i)^{-1} \int_\Gamma z^{1/2} \mathcal{R}_z(L_k^2)\, dz \qquad (9.39)$$

where $\mathcal{R}_z(L_k^2) \equiv (z - L_k^2)^{-1}$ is the resolvant and Γ is a simple closed curve in the right half (complex) plane which encloses the spectrum of L_k^2. Let $\kappa_1, \kappa_2 > 0$ be such that the eigenvalues of L_k^2 and $\hat{A}_k^2$ are in the interval $[2\kappa_1, \kappa_2]$. Here, we will take Γ as follows:

$$\Gamma = \{(\kappa_1, y) | y \in [-\kappa_1, \kappa_1]\} \cup \{(t,t) | t \in [\kappa_1, 2\kappa_2]\}$$
$$\cup \ \{(t, -t) | t \in [\kappa_1, 2\kappa_2]\} \cup \{(2\kappa_2, y) | y \in [-2\kappa_2, 2\kappa_2]\}.$$

Using an expression similar to (9.39) for $\hat{A}_k$ gives

$$L_k - \hat{A}_k = (2\pi i)^{-1} \int_\Gamma z^{1/2} \mathcal{R}_z(L_k^2)(L_k^2 - \hat{A}_k^2)\mathcal{R}_z(\hat{A}_k^2)\, dz. \qquad (9.40)$$

To estimate (9.40) we shall use the bounds given in the following lemma.

Lemma 9.7 *Let S and T be symmetric positive definite operators on $\mathcal{M}_k$ satisfying*

$$2\kappa_1 \|\chi\|_k^2 \leq (S^2\chi, \chi)_k \leq \kappa_2 \|\chi\|_k^2 ,$$
$$2\kappa_1 \|\chi\|_k^2 \leq (T^2\chi, \chi)_k \leq \kappa_2 \|\chi\|_k^2 ,$$

for all $\chi \in \mathcal{M}_k$. Assume that $\kappa_1 \geq c$ independently of k. We allow S, T and κ_2 to depend on k. Then

$$\int_\Gamma |z|^{1/2} \left\|S \cdot \mathcal{R}_z(S^2)\right\|_k \left\|\mathcal{R}_z(T^2)\right\|_k d|z| \lesssim (1 + \log(\kappa_2/\kappa_1)), \tag{9.41}$$

and for any $\chi \in \mathcal{M}_k$,

$$\int_\Gamma |z|^{1/2} \left\|S^{1/2} \cdot \mathcal{R}_z(S^2)\chi\right\|_k^2 d|z| \lesssim \|\chi\|_k^2 . \tag{9.42}$$

Proof. By symmetry, it suffices to derive the above bounds for the curve $\Gamma_+ \equiv \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where $\Gamma_1 \equiv \{(\kappa_1, y)|y \in [0, \kappa_1]\}$, $\Gamma_2 \equiv \{(t,t)|t \in [\kappa_1, 2\kappa_2]\}$ and $\Gamma_3 \equiv \{(2\kappa_2, y)|y \in [0, 2\kappa_2]\}$. By expansion in terms of eigenvectors, it is easy to see that

$$\left\|S^\beta \mathcal{R}_z(S^2)\right\|_k \leq \max_{\lambda \in [\sqrt{2\kappa_1}, \sqrt{\kappa_2}]} \lambda^\beta |\lambda^2 - z|^{-1} \qquad , \beta = 0, 1/2, 1. \tag{9.43}$$

A similar inequality obviously holds for T.

Let

$$\mathcal{F}_1(\gamma) \equiv \int_\gamma |z|^{1/2} \left\|S \cdot \mathcal{R}_z(S^2)\right\|_k \left\|\mathcal{R}_z(T^2)\right\|_k d|z|$$

and

$$\mathcal{F}_2(\gamma) \equiv \int_\gamma |z|^{1/2} \left\|S^{1/2} \cdot \mathcal{R}_z(S^2)\chi\right\|_k^2 d|z|.$$

Then by (9.43) and elementary estimates,

$$\begin{aligned}
\mathcal{F}_1(\Gamma_1) &\lesssim \int_{\Gamma_1} |z|^{1/2} \kappa_1^{-3/2} d|z| \lesssim 1, \\
\mathcal{F}_1(\Gamma_2) &\lesssim \int_{\Gamma_2} |z|^{-1} d|z| \lesssim \log(2\kappa_2/\kappa_1), \\
\mathcal{F}_1(\Gamma_3) &\lesssim \int_0^\infty \frac{\kappa_2}{\kappa_2^2 + y^2} dy \lesssim 1.
\end{aligned}$$

This verifies (9.41). Similar arguments give

$$\mathcal{F}_2(\Gamma_1) \lesssim (\int_{\Gamma_1} |z|^{1/2}\,\kappa_1^{-3/2}\,d|z|)\,\|\chi\|_k^2 \lesssim \|\chi\|_k^2\,,$$

$$\mathcal{F}_2(\Gamma_3) \lesssim (\int_0^\infty \frac{\kappa_2}{\kappa_2^2+y^2}\,dy)\,\|\chi\|_k^2 \lesssim \|\chi\|_k^2\,.$$

To bound $\mathcal{F}_2(\Gamma_2)$, we expand in terms of the eigenvectors of S. Let $\{\lambda_i, \theta_i\}$ denote the eigenvalue-eigenvector pairs for the operator S. Without loss of generality, we may assume that $\{\theta_i\}$ form an orthornormal basis for $\mathcal{M}_j$. Clearly, $2\kappa_1 \le \lambda_i^2 \le \kappa_2$ holds for each i. Decomposing

$$\chi = \sum_i c_i\theta_i$$

gives

$$\left\|S^{1/2}\cdot\mathcal{R}_z(S^2)\chi\right\|_k^2 = \sum_i \frac{\lambda_i c_i^2}{|\lambda_i^2 - z|^2}.$$

Integrating term by term yields,

$$\mathcal{F}_2(\Gamma_2) = \sum_i 2^{3/4} c_i^2 \int_{\kappa_1}^{2\kappa_2} \frac{\lambda_i t^{1/2}}{(\lambda_i^2 - t)^2 + t^2}\,dt. \tag{9.44}$$

Elementary manipulations show that the integrals in (9.44) are bounded uniformly in κ_1, κ_2, and λ_i. Hence $\mathcal{F}_2(\Gamma_2) \lesssim \|\chi\|_k^2$. This completes the proof of the lemma. ■

We now state and prove a lemma for estimating $L_k - \hat{A}_k$.

Lemma 9.8 *Let h_1 be sufficiently small. Then,*

$$\|L_k\chi - \hat{A}_k\chi\|_{H^1} \lesssim h_k^{\alpha-1}(1 + |\log h_k|)\left\|\hat{A}_k\chi\right\|_k$$

and

$$\left\|L_k\chi - \hat{A}_k\chi\right\|_k \lesssim h_k^{\alpha-1}\|\chi\|_{H^1}.$$

Proof. By Lemma 9.5,

$$\|L_k\chi - \hat{A}_k\chi\|_{H^1} \lesssim \left\|L_k^{1/2}(L_k - \hat{A}_k)\chi\right\|_k .$$

By (9.40), for any $\chi, \theta \in \mathcal{M}_k$,

$$\left(L_k^{1/2}(L_k - \hat{A}_k)\chi, \theta\right)_k = (2\pi i)^{-1} \int_\Gamma z^{1/2}(E_k\mathcal{R}_z(\hat{A}_k^2)\hat{A}_k\chi, L_k\mathcal{R}_z(L_k^2)\theta)_k\, dz$$

where

$$E_k = L_k^{-1/2}(L_k^2 - \hat{A}_k^2)\hat{A}_k^{-1}.$$

By the Schwartz inequality and (9.41), with $S = L_k$ and $T = \hat{A}_k$,

$$\left|\left(L_k^{1/2}(L_k - \hat{A}_k)\chi, \theta\right)_k\right| \lesssim (1 + |\log h_k|)\, \|E_k\|_k \left\|\hat{A}_k\chi\right\|_k \|\theta\|_k .$$

Note that we have used the fact that κ_1 is bounded uniformly from below and by (**A9.3**), we can take $\kappa_2 \lesssim h_k^{-4}$. Similarly, by (9.40),

$$\left(L_k\chi - \hat{A}_k\chi, \theta\right)_k = (2\pi i)^{-1} \int_\Gamma z^{1/2}(E_k\hat{A}_k^{1/2}\mathcal{R}_z(\hat{A}_k^2)\hat{A}_k^{1/2}\chi, L_k^{1/2}\mathcal{R}_z(L_k^2)\theta)_k\, dz.$$

By the Schwartz inequality, Lemma 9.5 and (9.42),

$$\left|\left(L_k\chi - \hat{A}_k\chi, \theta\right)_k\right| \lesssim \|E_k\|_k\, \|\chi\|_{H^1}\, \|\theta\|_k .$$

Thus, the proof of the lemma will be complete if we can show that

$$\|E_k\|_k \lesssim h_k^{\alpha-1}.$$

Obviously,

$$L_k^2 - \hat{A}_k^2 = \hat{A}_k D_k + D_k^t\hat{A}_k + D_k^t D_k$$

and hence

$$\|E_k\|_k \le \left\|L_k^{-1/2}\hat{A}_k D_k \hat{A}_k^{-1}\right\|_k + \left\|L_k^{-1/2}D_k^t \hat{A}_k \hat{A}_k^{-1}\right\|_k + \left\|L_k^{-1/2}D_k^t D_k \hat{A}_k^{-1}\right\|_k \tag{9.45}$$

Using Lemmas 9.5 and 9.6 and (9.6) gives

$$\left\|L_k^{-1/2}D_k^t \hat{A}_k \hat{A}_k^{-1}\right\|_k = \left\|D_k L_k^{-1/2}\right\|_k = \left\|\Pi_k D L_k^{-1/2}\right\|_k \lesssim 1. \tag{9.46}$$

Similarly,

$$\left\|L_k^{-1/2}D_k^t D_k \hat{A}_k^{-1}\right\|_k \lesssim \left\|D_k L_k^{-1/2}\right\|_k \left\|D_k \hat{A}_k^{-1/2}\right\|_k \lesssim 1.$$

For the first term of (9.45), using Lemma 9.5 gives

$$\left\|L_k^{-1/2}\hat{A}_k D_k \hat{A}_k^{-1}\right\|_k \le \left\|L_k^{-1/2}\hat{A}_k^{1/2}\right\|_k \left\|\hat{A}_k^{1/2} D_k \hat{A}_k^{-1}\right\|_k \lesssim \left\|\hat{A}_k^{1/2} D_k \hat{A}_k^{-1}\right\|_k.$$

Combining the above estimates, making an obvious change of variable and applying Lemma 9.6 implies that the proof of the lemma will be complete if we show

$$\|D_k\chi\|_{H^1} \lesssim h_k^{\alpha-1}\left\|\hat{A}_k\chi\right\|_k \quad \forall\chi\in\mathcal{M}_k. \tag{9.47}$$

Fix $\chi \in \mathcal{M}_k$ and let $w \in \mathcal{M}$ be the solution to

$$\hat{A}(w,\phi) = \left(\hat{A}_k\chi,\phi\right) \quad \forall\phi\in\mathcal{M}.$$

Clearly $\chi = \hat{P}_k w$. Now

$$\|D_k\chi\|_{H^1} \le \|\Pi_k D(\chi - w)\|_{H^1} + \|\Pi_k D w\|_{H^1}.$$

Applying (9.6), (**A9.3**), (**A9.4**), and Lemma 9.6 gives

$$\|\Pi_k D(\chi - w)\|_{H^1} \lesssim h_k^{-1}\|\chi - w\|_{H^1} \le h_k^{\alpha-1}\|w\|_{H^{1+\alpha}}.$$

Finally, by (**A9.4**), Lemma 9.6 and (9.7),

$$\|\Pi_k D w\|_{H^1} \lesssim h_k^{\alpha-1}\|\Pi_k D w\|_{H^\alpha} \lesssim h_k^{\alpha-1}\|Dw\|_{H^\alpha} \lesssim h_k^{\alpha-1}\|w\|_{1+\alpha}.$$

Inequality (9.47) now follows combining the above estimates with (**A9.1**). This completes the proof of Lemma 9.8 ■

We can now prove Lemma 9.3.

Proof of Lemma 9.3

By (9.38), Lemma 9.5 and Proposition 9.2, it suffices to show that

$$\left\|G_k^t \chi\right\|_k \lesssim h^{-1/2-\epsilon} \|\chi\|_{H^1}.$$

In turn, by Lemma 9.8 and the triangle inequality, noting that $\alpha > 1/2$, it suffices to show

$$\left\|D_k^t \chi\right\|_k \lesssim h^{-1/2-\epsilon} \|\chi\|_{H^1}. \tag{9.48}$$

Let $\theta \in \mathcal{M}_k$. Then by (**A9.2**), (**A9.4**) and Lemma 9.5

$$\left(D_k^t \chi, \theta\right)_k = \left(D^t \chi, \theta\right) \lesssim \|\chi\|_{H^1} \|\theta\|_{H^{1/2+\epsilon}} \lesssim h_k^{-1/2-\epsilon} \|\chi\|_{H^1} \|\theta\|_k .$$

Inequality (9.48) immediately follows. This completes the proof of the lemma.

We shall need two additional lemmas for the proof of Lemma 9.4. The first involves stability and approximation for the operator I_k^{k-1}.

Lemma 9.9 *For all $\chi \in \mathcal{M}_k$,*

$$\|(I - I_k^{k-1})\chi\|_H \lesssim h_k \|\chi\|_{H^1} \tag{9.49}$$

and

$$\|I_k^{k-1} \chi\|_{H^1} \lesssim \|\chi\|_{H^1}. \tag{9.50}$$

Proof. Note that by (9.8) and (9.9), for $\varphi \in \mathcal{M}_{k-1}$,

$$((I_k^{k-1} - \Pi_k)\chi, \varphi)_{k-1} = (\chi, \varphi)_k - (\chi, \varphi) \lesssim h_k \|\chi\|_{H^1} \|\varphi\|_{k-1} .$$

This implies that

$$\|(I_k^{k-1} - P_{k-1}^0)\chi\|_{k-1} \lesssim h_k \|\chi\|_{H^1}.$$

The lemma then follows from Lemmas 9.5 and 9.6 and (**A9.4**). ■

Lemma 9.10 *For all* $\chi \in \mathcal{M}_{k-1}$,

$$\left\|\hat{A}_k\chi\right\|_k \lesssim h_k^{\alpha-1}\left\|\hat{A}_{k-1}\chi\right\|_{k-1}, \tag{9.51}$$

$$\|L_k\chi\|_k \lesssim h_k^{\alpha-1}\|L_{k-1}\chi\|_{k-1}, \tag{9.52}$$

$$\|\hat{A}_{k-1}\chi\|_{k-1} \lesssim k^{\alpha-1}\left\|I_k^{k-1}L_k\chi\right\|_{k-1} \tag{9.53}$$

and

$$\|L_k\chi\|_k \lesssim h_k^{2\alpha-2}\left\|I_k^{k-1}L_k\chi\right\|_{k-1}. \tag{9.54}$$

Proof. By Proposition 9.2, Lemma 9.5, (**A9.3**) and (**A9.4**), for all $\varphi \in \mathcal{M}_k$,

$$\left\|\varphi - \hat{P}_{k-1}\varphi\right\|_k \lesssim h_k^{\alpha}\|\hat{\varphi}\|_{H^1} \lesssim h_k^{\alpha-1}\|\varphi\|_k$$

hence

$$\left\|\hat{P}_{k-1}\varphi\right\|_k \lesssim h_k^{\alpha-1}\|\varphi\|_k.$$

Therefore, for $\chi \in \mathcal{M}_{k-1}$,

$$\begin{aligned}\left(\hat{A}_k\chi,\varphi\right)_k &= \hat{A}(\chi,\varphi) = \hat{A}\left(\chi,\hat{P}_{k-1}\varphi\right)\\ &= \left(\hat{A}_{k-1}\chi,\hat{P}_{k-1}\varphi\right)_{k-1} \lesssim h_k^{\alpha-1}\left\|\hat{A}_{k-1}\chi\right\|_{k-1}\|\varphi\|_k.\end{aligned}$$

This proves (6.20). Inequality (9.52) then follows from (6.20) and Lemma 9.5.

We next prove (9.53). Noting that $\hat{A}_{k-1} = I_k^{k-1}\hat{A}_k$, the triangle inequality and Lemma 9.8 give

$$\begin{aligned}\left\|\hat{A}_{k-1}\chi\right\|_{k-1} &= \left\|I_k^{k-1}\hat{A}_k\chi\right\|_{k-1} \le (\left\|I_k^{k-1}L_k\chi\right\|_{k-1} + \left\|(\hat{A}_k - L_k)\chi\right\|_k)\\ &\lesssim h_k^{\alpha-1}(\left\|I_k^{k-1}L_k\chi\right\|_{k-1} + \|\chi\|_{H^1}).\end{aligned}$$

Finally, we note that by Lemma 9.5 and (9.8),

$$\|\chi\|_{H^1} \lesssim (L_k\chi,\chi)_k \lesssim \left\|I_k^{k-1}L_k\chi\right\|_{k-1}\|\chi\|_{H^1}$$

and hence

$$\|\chi\|_{H^1} \lesssim \left\| I_k^{k-1} L_k \chi \right\|_{k-1}.$$

Combining the above inequalities completes the proof of (9.53). Inequality (9.54) follows immediately from (9.53), (6.20) and Lemma 9.5. ■

We are now ready to prove Lemma 9.4. However, before doing so, we note a few properties of our operators which are immediate consequences of the defining relations. As noted earlier, $\hat{A}_{k-1} = I_k^{k-1}\hat{A}_k$. Similarly $D_{k-1} = I_k^{k-1} D_k$. In addition, the operator I_k^{k-1} is symmetric on both $\mathcal{M}_k$ with the $(\cdot,\cdot)_k$ inner product as well as $\mathcal{M}_{k-1}$ with the $(\cdot,\cdot)_{k-1}$ inner product.

Proof of Lemma 9.4

For $\chi \in \mathcal{M}_{k-1}$,

$$\left| \|\!|\chi|\!\|_{k,1}^2 - \|\!|\chi|\!\|_{k-1,1}^2 \right| = |((\widetilde{L}_{k-1} - L_{k-1})\chi, \chi)_{k-1}|,$$

where $\widetilde{L}_{k-1} \equiv I_k^{k-1} L_k$. Note that the operator $\widetilde{L}_{k-1} : \mathcal{M}_{k-1} \mapsto \mathcal{M}_{k-1}$ is symmetric and the eigenvalues of $\widetilde{L}_{k-1}^2$ are in the interval $[c, Ch_k^{-4}]$ for appropriate constants c and C (independent of k). Applying an expression analogous to (9.40) gives

$$\left((L_{k-1} - \widetilde{L}_{k-1})\chi, \chi\right)_{k-1} = (2\pi i)^{-1} \int_\Gamma z^{1/2} \left(F_k L_{k-1} \mathcal{R}_z(L_{k-1}^2)\chi, \widetilde{L}_{k-1}\mathcal{R}_z(\widetilde{L}_{k-1}^2)\chi\right)_{k-1} dz$$

where

$$F_k = \widetilde{L}_{k-1}^{-1}(L_{k-1}^2 - \widetilde{L}_{k-1}^2)L_{k-1}^{-1}.$$

By the Schwartz inequality and (9.42),

$$\begin{aligned} \left| \left((L_{k-1} - \widetilde{L}_{k-1})\chi, \chi\right)_{k-1} \right| &\lesssim \|F_k\|_{k-1} \|L_{k-1}^{1/2}\chi\|_{k-1} \|\widetilde{L}_{k-1}^{1/2}\chi\|_{k-1} \\ &\lesssim \|F_k\|_{k-1} \|\!|\chi|\!\|_{k,1}^2 \end{aligned}$$

where the second inequality follows from Lemma 9.5 and the identity $(\widetilde{L}_{k-1}\chi, \chi)_{k-1} = (L_k\chi, \chi)_k$. To complete the proof of the lemma, we need only bound $\|F_k\|_{k-1}$.

We start first with the identity

$$F_k = Q_1 + Q_2 + Q_3$$

where

$$\begin{aligned} Q_1 &= (I - I_k^{k-1})(L_k - \hat{A}_k)L_{k-1}^{-1}, \\ Q_2 &= \widetilde{L}_{k-1}^{-1} I_k^{k-1}(L_k - \hat{A}_k)(I - I_k^{k-1})\hat{A}_k L_{k-1}^{-1} \\ Q_3 &= \widetilde{L}_{k-1}^{-1}(L_{k-1}^2 - I_k^{k-1}L_k^2 - \hat{A}_{k-1}^2 + I_k^{k-1}\hat{A}_k^2)L_{k-1}^{-1}. \end{aligned}$$

Obviously, it suffices to bound the norms $\|Q_i\|_{k-1}$, for $i = 1, 2, 3$.

Let $\chi, \theta \in \mathcal{M}_{k-1}$. For Q_1, by Lemmas 9.5, 9.8, 9.9 and 9.10, we have

$$\begin{aligned} \|Q_1\chi\|_{k-1} &\lesssim h_k\|(L_k - \hat{A}_k)L_{k-1}^{-1}\chi\|_{H^1} \lesssim h_k^{\alpha}(1 + |\log h_k|) \left\|L_k L_{k-1}^{-1}\chi\right\|_k \\ &\lesssim h_k^{2\alpha-1}(1 + |\log h_k|)\|\chi\|_{k-1}. \end{aligned}$$

For Q_2, we have

$$\begin{aligned} |(Q_2\chi, \theta)_{k-1}| &= \left|\left(\hat{A}_k L_{k-1}^{-1}\chi, (I - I_k^{k-1})(L_k - \hat{A}_k)\widetilde{L}_{k-1}^{-1}\theta\right)_k\right| \\ &\le \left\|\hat{A}_k L_{k-1}^{-1}\chi\right\|_k \left\|(I - I_k^{k-1})(L_k - \hat{A}_k)\hat{A}_k^{-1}\right\|_k \left\|\hat{A}_k\widetilde{L}_{k-1}^{-1}\theta\right\|_k. \end{aligned}$$

Thus, applying Lemmas 9.5, 9.8, 9.9, and 9.10 gives

$$\|Q_2\|_{k-1} \lesssim h_k^{4\alpha-3}(1 + |\log h_k|).$$

For Q_3, we obviously have

$$\|Q_3\|_{k-1} \le \|Q_{3,1}\|_{k-1} + \|Q_{3,2}\|_{k-1} + \|Q_{3,3}\|_{k-1}$$

where

$$\begin{aligned} Q_{3,1} &= \widetilde{L}_{k-1}^{-1}\hat{A}_{k-1}(I_k^{k-1} - I)D_k L_{k-1}^{-1} \\ Q_{3,2} &= \widetilde{L}_{k-1}^{-1}(D_{k-1}^t\hat{A}_{k-1} - I_k^{k-1}D_k^t\hat{A}_k)L_{k-1}^{-1} \\ Q_{3,3} &= \widetilde{L}_{k-1}^{-1}(D_{k-1}^t D_{k-1} - I_k^{k-1}D_k^t D_k)L_{k-1}^{-1}. \end{aligned}$$

For $Q_{3,1}$, by (9.47), (9.53), and Lemma 9.9,

$$\begin{aligned}\|Q_{3,1}\chi\|_{k-1} &\lesssim h_k\|\widetilde{L}_{k-1}^{-1}\hat{A}_{k-1}\|_{k-1}\|D_kL_{k-1}^{-1}\chi\|_{H^1}\\ &\lesssim h^{3\alpha-2}\|\chi\|_{k-1}.\end{aligned}$$

For $Q_{3,2}$,

$$\begin{aligned}|(Q_{3,2}\chi,\theta)_{k-1}| &= \left|\left(\hat{A}_kL_{k-1}^{-1}\chi,(I-I_k^{k-1})D_k\widetilde{L}_{k-1}^{-1}\theta\right)_k\right|\\ &\leq \left\|\hat{A}_kL_{k-1}^{-1}\chi\right\|_k\left\|(I-I_k^{k-1})D_k\hat{A}_k^{-1}\right\|_k\left\|\hat{A}_k\widetilde{L}_{k-1}^{-1}\theta\right\|_k.\end{aligned}$$

Applying (9.47) and Lemmas 9.9 and 9.10 gives

$$\|Q_{3,2}\|_{k-1}\lesssim h_k^{4\alpha-3}.$$

Finally, for $Q_{3,3}$,

$$\begin{aligned}|(Q_{3,3}\chi,\theta)_{k-1}| &= \left|\left(D_kL_{k-1}^{-1}\chi,(I-I_k^{k-1})D_k\widetilde{L}_{k-1}^{-1}\theta\right)_k\right|\\ &\leq \left\|D_kL_k^{-1}\right\|_k\left\|L_kL_{k-1}^{-1}\chi\right\|_k\left\|(I-I_k^{k-1})D_k\hat{A}_k^{-1}\right\|_k\left\|\hat{A}_k\widetilde{L}_{k-1}^{-1}\theta\right\|_k.\end{aligned}$$

Applying (9.46), (9.47), and Lemmas 9.5, 9.9 and 9.10 gives

$$\|Q_{3,2}\|_{k-1}\lesssim h_k^{4\alpha-3}.$$

Combining the above inequalities proves Lemma 9.4.

Chapter 10

Parallel Multilevel Preconditioners

In the last three chapters, we have studied the multigrid algorithms that consist of fine level smoothing and coarse level correction. In this chapter, we will take a different approach to make use of the multilevel structure to design another kind of efficient algorithm. Namely we will construct preconditioners from multilevel spaces.

The idea is somehow related to the so-called hierarchical basis preconditioners (c.f. [94, 11]). Roughly speaking, in that method, the finest space is decomposed as a "nearly orthogonal" summation of some coarse spaces via the interpolation operators. But this method degrades in three or higher dimensions. In contrast, our method is based on the decomposition via L^2 projections onto coarse level spaces. This method works equally well in any number of dimensions and most importantly it can be implemented in a completely parallel fashion.

We shall study this new technique for developing preconditioners first in an abstract setting and then by considering applications to second order elliptic problems. In Section 10.1, we will set up an abstract framework of our theory. The condition numbers of the proposed preconditioners will be estimated in terms of a number of a priori assumptions. The results in this section will be enough for the applications given in Section 10.3 and 10.4. In Section 10.2, we will make more investigations of the theory developed in Section 10.1. In particular, the motivations will be explained and other applications will be discussed. In Section 10.3, the theory is applied to second order elliptic boundary value problem with finite element discretizations and the parallel as well as serial complexity of the resulting algorithms is discussed. Finally, analysis of an algorithm applying to an interface problem is given in Section 10.4.

10.1 Abstract Theory

An abstract framework for the construction of parallel multilevel preconditioners will be given in this section. The preconditioners will be proposed in terms of a certain multilevel structure and estimates for relevant condition numbers will be obtained based on a number of a priori assumptions. In this abstract setting, we shall not be concerned with any implementation or complexity issues. The results in this section will be enough for applications in Section 10.3 and 10.4. More theoretical analysis and discussion of other possible implications of the abstract framework can be found in the next section.

Given a Hilbert space $\mathcal{M}$ and two inner products $(\cdot,\cdot)$ and $A(\cdot,\cdot)$, we are interested in solving the following problem on $\mathcal{M}$:

$$A(u,v) = (f,v), \quad \forall v \in \mathcal{M}. \tag{10.1}$$

where $f \in \mathcal{M}$ is given.

Our purpose is to develop some preconditioning methods for the given problem by using certain multilevel structure. The main ingredient in our theory is a hierarchy of Hilbert spaces as follows:

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_j \tag{10.2}$$

where $\mathcal{M}_j \equiv \mathcal{M}$.

For each $k = 1, \ldots, j$, we introduce the operators $P_k, Q_k : \mathcal{M} \longrightarrow \mathcal{M}_k$ defined, for all $u \in \mathcal{M}, v \in \mathcal{M}_k$, by

$$\begin{aligned} A(P_k u, v) &= A(u, v), \\ (Q_k u, v) &= (u, v), \end{aligned}$$

and the operators $A_k : \mathcal{M}_k \longrightarrow \mathcal{M}_k$ defined by

$$(A_k u, v) = A(u, v) \qquad \forall u, v \in \mathcal{M}_k.$$

If $k = j$, we denote $A \stackrel{\text{def}}{=} A_j$, by definition, the problem (10.1) is equivalent to the following:

$$Au = f.$$

The goal of this section is to propose and study the following preconditioner:

$$B = A_1^{-1} Q_1 + \sum_{k=2}^{j} \lambda_k{}^{-1} Q_k \tag{10.3}$$

where $\lambda_k = \rho(A_k)$, the spectral radius of A_k.

The motivation of constructing B in such a way is not trivial, and will be discussed in the forthcoming section. In the current section, we will derive some important properties of B in terms of those of A.

We will show that B is a good preconditioner of A by giving estimates of the condition number of BA based on a number of a priori assumptions. Namely, we will prove inequalities of the following type:

$$c_j A(v, v) \le A(BAv, v) \le C_j A(v, v). \tag{10.4}$$

We will relate the constants c_j and C_j to j and the constants appearing in the subsequent a priori assumptions. The following relations, which are obtained directly from definitions, are useful in our analysis below.

$$Q_k A = A_k P_k \tag{10.5}$$

$$Q_k Q_l = Q_l Q_k = Q_l \text{ if } l \le k. \tag{10.6}$$

From (10.5), we can easily see that B is symmetric positive definite and the following holds:

$$A(BAv, v) = A(P_1 v, P_1 v) + \sum_{k=2}^{j} \lambda_k^{-1} \|A_k P_k v\|^2. \tag{10.7}$$

The estimate of C_j in (10.4) straight forward without any additional assumption:

Lemma 10.1

$$A(BAv, v) \le jA(v, v), \quad \forall v \in \mathcal{M}.$$

Proof. This follows directly from (10.7). Thus

$$A(BAv, v) \le \sum_{k=1}^{j} A(P_k v, P_k v) \le jA(v, v),$$

since P_k is a projection. ■

Next we will estimate c_j in (10.4) under some additional a priori assumptions. The first result will be based on two assumptions regarding the approximation and stability property of the projection Q_k.

(A10.1) $\|(I - Q_{k-1})v\|^2 \le K_1 \lambda_k^{-1} A(v, v), \quad \forall v \in \mathcal{M}_k.$

(A10.2) $A(Q_k v, Q_k v) \le K_2 A(v, v), \quad \forall v \in \mathcal{M}.$

Here the constants K_1, K_2 may moderately depend on j in some applications. For convenience, we will assume that $K_1 \ge 1$.

Lemma 10.2 *Under the assumptions (***A10.1***) and (***A10.2***)*

$$A(v, v) \le K_1 K_2 j A(BAv, v), \quad \forall v \in \mathcal{M}.$$

Proof. Writing $v = Q_1 v + \sum_{k=2}^{j}(Q_k - Q_{k-1})v$, we have

$$\begin{aligned}
A(v,v) &= A(Q_1 v, v) + \sum_{k=2}^{j} A((I - Q_{k-1})Q_k v, v) \\
&\leq A(Q_1 v, P_1 v) + \sum_{k=2}^{j}((I - Q_{k-1})Q_k v, A_k P_k v) \\
&\leq A(Q_1 v, Q_1 v)^{\frac{1}{2}} A(P_1 v, P_1 v)^{\frac{1}{2}} + \sum_{k=2}^{j}(K_1 \lambda_k^{-1} A(Q_k v, Q_k v))^{\frac{1}{2}} \|A_k P_k v\| \\
&\leq (K_2 A(v,v))^{\frac{1}{2}} A(P_1 v, P_1 v)^{\frac{1}{2}} + \sum_{k=2}^{j}(K_1 K_2 A(v,v))^{\frac{1}{2}} \lambda_k{}^{-\frac{1}{2}} \|A_k P_k v\| \\
&\leq (K_2 + (j-1)K_1 K_2)^{\frac{1}{2}} A(v,v)^{\frac{1}{2}} (A(P_1 v, P_1 v) + \sum_{k=2}^{j} \lambda_k^{-1} \|A_k P_k v\|^2)^{\frac{1}{2}}.
\end{aligned}$$

The desired result then follows. ∎

The second result for estimating c_j in (10.4) will be based on an approximation assumption of the projection P_k, namely

(A10.3) $\quad A((I - P_{k-1})v, v) \leq K_3^{\alpha}(\lambda_k^{-1}\|A_k v\|^2)^{\alpha} A(v,v)^{1-\alpha}, \quad \forall v \in \mathcal{M}_k.$

Here $\alpha \in (0,1]$ is a constant and again we assume $K_3 \geq 1$.

Lemma 10.3 *Under the assumption (***A10.3***),*

$$A(v,v) \leq K_3 j^{\frac{1}{\alpha}-1} A(BAv, v), \quad \forall v \in \mathcal{M}.$$

Proof. Writing $v = P_1 v + \sum_{k=2}^{j}(I - P_{k-1})P_k v$ and using Hölder's inequality, we can deduce that

$$A(v,v) = A(P_1 v, P_1 v) + \sum_{k=2}^{j} A((I - P_{k-1})P_k v, v)$$

$$\leq \quad A(P_1v, P_1v)^\alpha A(v,v)^{1-\alpha} + \sum_{k=2}^{j} K_3^\alpha (\lambda_k^{-1}\|A_kP_kv\|^2)^\alpha (Av,v)^{1-\alpha}$$

$$\leq \quad (A(P_1v, P_1v) + K_3 \sum_{k=2}^{j} \lambda_k{}^{-1}\|A_kP_ku\|^2)^\alpha j^{1-\alpha} A(v,v)^{1-\alpha}$$

The proof is then completed by dividing by $A(v,v)^{1-\alpha}$. ■

In summary, we have

Theorem 10.1 *Under assumptions (***A10.1***) and (***A10.2***),*

$$(K_1K_2j)^{-1}A(v,v) \leq A(BAv,v) \leq jA(v,v), \quad \forall v \in \mathcal{M}.$$

Hence

$$\kappa(BA) \leq K_1K_2j^2.$$

*Under the assumption (***A10.3***),*

$$K_3^{-1}j^{1-\frac{1}{\alpha}}A(v,v) \leq A(BAv,v) \leq jA(v,v), \quad \forall v \in \mathcal{M}.$$

Hence

$$\kappa(BA) \leq K_3 j^{\frac{1}{\alpha}}.$$

We can avoid computing the action of A_1^{-1} by modifying B as follows:

$$\tilde{B} \stackrel{\text{def}}{=} \sum_{k=1}^{j} \lambda_k^{-1} Q_k. \tag{10.8}$$

In this case, the assumptions (**A10.1**) and (**A10.2**) can be replaced by the following:

(A10.1′) $\quad \|(I - Q_{k-1})v\|^2 \leq K_1' \lambda_k^{-1} A(v,v), \quad \forall v \in \mathcal{M}.$

Analogous to Theorem 10.1, we have

Theorem 10.2

$$\kappa(\tilde{B}A) \le \begin{cases} (\kappa(A_1) + K_1K_2j)j & \text{if } (\mathbf{A10.1}) \text{ and } (\mathbf{A10.2}) \text{ hold;} \\ (\kappa(A_1) + K_1'j)j & \text{if } (\mathbf{A}10.1') \text{ holds;} \\ (\kappa(A_1)^{\frac{\alpha}{1-\alpha}} + K_3^{\frac{\alpha}{1-\alpha}}(j-1))^{\frac{\alpha}{1-\alpha}} j^{\frac{1-\alpha}{\alpha}} & \text{if } (\mathbf{A10.3}) \text{ holds.} \end{cases}$$

Remark 10.1 The estimates given in above theorem suggest that under the assumptions (**A10.1**) and (**A10.2**) for example, if the magnitude of $\kappa(A_1)$ is camparable with K_1K_2j, the preconditioner $\widetilde{B}$ would be as good as B. The last estimate in the above theorem looks a little clumsy. In fact it can be replaced by

$$\kappa(\widetilde{B}A) \le \max(\kappa(A_1), K_3)j^{\frac{1}{\alpha}}, \quad \text{if } (\mathbf{A10.3}) \text{ holds.}$$

Remark 10.2 By a telescope argument, it is not hard to see that (**A**10.1′) is actually a consequence of (**A10.1**) and (**A10.2**) if furthermore λ_k's satisfy certain growth condition as follows:

(A10.4) $\exists \sigma > 0$ such that : $\lambda_{k+1} \ge (1+\sigma)\lambda_k, \quad \forall k = 1, 2, \cdots, j-1.$

In fact, if both (**A10.1**) and (**A10.2**) are satisfied, we have

$$\begin{aligned} \|(I - Q_{k-1})v\| &\le \sum_{l=k}^{j} \|(I - Q_{l-1})Q_l v\| \\ &\le K_1^{\frac{1}{2}} \sum_{l=k}^{j} \lambda_l^{-\frac{1}{2}} A(Q_l v, Q_l v)^{\frac{1}{2}} \\ &\le (K_1K_2)^{\frac{1}{2}} \left(\sum_{l=k}^{j} \lambda_l^{-\frac{1}{2}} \right) A(v,v)^{\frac{1}{2}} \\ &\le (K_1K_2)^{\frac{1}{2}} \frac{\sqrt{\sigma+1}}{\sqrt{\sigma+1}-1} A(v,v)^{\frac{1}{2}}. \end{aligned}$$

Therefore, ($\mathbf{A}10.1'$) holds with

$$K_1' = \frac{2(1+\sigma)(2+\sigma)}{\sigma^2} K_1 K_2.$$

10.2 More on the Abstract Theory

In the preceding section, we have proposed and studied the preconditioner B given by (10.3). This section is devoted to more theoretical investigations of B. We will discuss how this preconditioner was motivated and developed. This section is a theoretical refinement of the last one and it has little to do with the applications in forthcoming sections. Some other possible applications will also be mentioned without giving much details.

Our discussion begins with the following identities:

$$(Q_k - Q_{k-1})Q_l = 0, \quad \text{if } l < k, \tag{10.9}$$

$$(Q_k - Q_{k-1})^2 = (Q_k - Q_{k-1}), \tag{10.10}$$

$$(Q_k - Q_{k-1})(Q_l - Q_{l-1}) = 0, \text{if } l \neq k. \tag{10.11}$$

These identities can be obtained directly from the definition of Q_k's and the inclusion property in (10.2). If we set $Q_0 = 0$, then in terms of Q_k's, , we have an orthogonal decomposition of $\mathcal{M}$ as follows:

$$\mathcal{M} = \bigoplus_{k=1}^{j} (Q_k - Q_{k-1})\mathcal{M}.$$

where $(Q_k - Q_{k-1})\mathcal{M}$ is the range of the operator $Q_k - Q_{k-1}$. Notice that

$$\mathcal{R}(Q_1) = \mathcal{M}_1, \quad (Q_k - Q_{k-1})\mathcal{M} = \mathcal{M}_{k-1}^{\perp} \text{ in } \mathcal{M}_k,$$

where $\perp$ is with respect to the inner product $(\cdot,\cdot)$.

It is straightforward to check, under the assumption (**A10.1**), that

$$A(v_k, v_k) \asymp \lambda_k \|v_k\|^2, \quad \forall v \in (Q_k - Q_{k-1})\mathcal{M}.$$

This means that on $(Q_k - Q_{k-1})\mathcal{M}$, the operator A behaviors like a constant λ_k spectrally.

Obviously, for $v \in \mathcal{M}$, we can write

$$Av = AQ_1 v + \sum_{k=2}^{j} A(Q_k - Q_{k-1})v, \tag{10.12}$$

but the restriction of A to $\mathcal{M}_1$ is A_1 and as mentioned just now A is spectrally equivalent to λ_k on $(Q_k - Q_{k-1})\mathcal{M}$, therefore the following expression should be spectrally close to (10.12):

$$\mathcal{A}v \stackrel{\text{def}}{=} A_1 Q_1 v + \sum_{k=2}^{j} \lambda_k (Q_k - Q_{k-1})v,$$

which defines an operator as follows

$$\mathcal{A} \stackrel{\text{def}}{=} A_1 Q_1 + \sum_{k=2}^{j} \lambda_k (Q_k - Q_{k-1})$$

with the corresponding bilinear form given by

$$\mathcal{A}(u, v) = A(Q_1 u, Q_1 v) + \sum_{k=2}^{j} \lambda_k ((Q_k - Q_{k-1})u, (Q_k - Q_{k-1})v), \quad u, v \in \mathcal{M}. \tag{10.13}$$

One would expect the operator $\mathcal{A}$ to be spectrally comparable to A, and, as a matter of fact, with a proof identical to that of Theorem 10.1, we can show

Theorem 10.3 *Under assumptions (**A10.1**) and (**A10.2**),*

$$j^{-1} A(v, v) \le \mathcal{A}(v, v) \le K_1 K_2 j A(v, v), \quad \forall v \in \mathcal{M}. \tag{10.14}$$

Because of the identities (10.9)-(10.11), we see that the operator $\mathcal{A}$ is a summation of a number of mutually orthogonal operators. This is a very important property of $\mathcal{A}$, as a consequence we have:

Theorem 10.4

$$\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A}^{-1} = A_1^{-1} Q_1 + \sum_{k=2}^{j} \lambda_k^{-1} (Q_k - Q_{k-1}). \tag{10.15}$$

More generally

$$\mathcal{A}^s = A_1^s Q_1 + \sum_{k=2}^{j} \lambda_k^s (Q_k - Q_{k-1}), \quad \forall s \in \mathbb{R}^1. \tag{10.16}$$

The identity (10.16) requires some explanation. There is no problem for any integer value s in which case (10.16) follows by induction. We note that $\mathcal{A}$ is a SPD operator on a finite dimensional space $\mathcal{M}$. For any intger n, there is a unique SPD $n-$th root of A, hence the verification for $s = \frac{1}{n}$ then follows by taking the $n-$th power of the right hand side of (10.16). Combining the above arguments, we then see that (10.16) also holds for any rational number s. Since any irrational number can be approximated by a sequence of rationals, (10.16) then also holds when s is an irrational number.

Theorem 10.5 *Under assumptions (***A10.1***) and (***A10.2***),*

$$j^{-s}(A^s v, v) \le (\mathcal{A}^s v, v) \le (K_1 K_2 j)^s (A^s v, v), \quad \forall v \in \mathcal{M}, \;\; s \in [0, 1]. \tag{10.17}$$

where $\mathcal{A}^s$ *is given by* (10.16).

Proof. For $s = 0$ (10.17) is trivial and for $s = 1$ (10.17) is just (10.14). Therefore the standard interpolation technique (c.f. [58]) gives the desired result for $s \in (0, 1)$. ■

Remark 10.3 Take $s = \frac{1}{2}$ in above theorem, we conclude that the square root of $\mathcal{A}$ is a good preconditioner for the square root of the discrete elliptic operator A, which is directly related to the $H^{\frac{1}{2}}$ norm. As a matter of fact, a new preconditioner via domain decomposition techniques can be constructed using these ideas and will be reported on in [27].

The relation between $\mathcal{B}$ defined by (10.15) and B defined by (10.3) is shown in the follwoing

Lemma 10.4 *Under the assumption (***A10.4***),*

$$(\mathcal{B}v, v) \leq (Bv, v) \leq (1 + \frac{1}{\sigma})(\mathcal{B}v, v) \tag{10.18}$$

where B is given by (10.3).

Proof. By definition of $\mathcal{B}$, we have

$$\begin{aligned}
(\mathcal{B}v, v) &= (A_1^{-1}Q_1 v, v) + \sum_{k=2}^{j} \lambda_k{}^{-1}((Q_k - Q_{k-1})v, v) \\
&= (A_1^{-1}Q_1 v, v) + \sum_{k=2}^{j} \lambda_k{}^{-1}((Q_k - Q_{k-1})v, v) \\
&= (A_1^{-1}Q_1 v, v) - \lambda_2^{-1}\|Q_1 v\|^2 \\
&\quad + \sum_{k=2}^{j-1}(\lambda_k{}^{-1} - \lambda_{k+1}^{-1})(Q_k v, v) + \lambda_j^{-1}(Q_j v, v) \\
&\geq (A_1^{-1}Q_1 v, v) - \frac{1}{1+\sigma}\lambda_1^{-1}\|Q_1 v\|^2 \\
&\quad + \frac{\sigma}{1+\sigma}\sum_{k=2}^{j-1} \lambda_k{}^{-1}(Q_k v, v) + \lambda_j^{-1}(Q_j v, v) \\
&\geq \frac{\sigma}{1+\sigma}(Bv, v).
\end{aligned}$$

Since the other part of the inequality is trivial, the proof is complete. ■

Remark 10.4 In general, without the assumption (**A10.4**), the following estimate can be obtained:

$$(\mathcal{B}v, v) \le (Bv, v) \le j(\mathcal{B}v, v). \tag{10.19}$$

In the next theorem, we consider the case of the sum of two operators. Let $\hat{A}(\cdot,\cdot)$ be another symmetric positive definite form and let $\hat{A}$, and $\{\hat{\lambda}_k\}$ be defined analogously in terms of $\hat{A}(\cdot,\cdot)$. Consider the operator $\bar{B} : \mathcal{M} \mapsto \mathcal{M}$ defined by

$$\bar{B} = \sum_{k=1}^{J} (\lambda_k + \hat{\lambda}_k)^{-1} (Q_k - Q_{k-1}).$$

Theorem 10.6 *Assume (**A**10.1$'$) holds for both A and $\hat{A}$. Then,*

$$\kappa(\bar{B}(A + \hat{A})) \le Cj^2$$

where the constant C may depend on $\kappa(A_1)$ and $\kappa(\hat{A}_1)$.

A direct application of above theorem is to the discrete systems which arise in parabolic time stepping algorithms. At each time level, a function $u^n \in \mathcal{M}$ satisfying

$$(I + \tau A)u^n = f^n,$$

with known $f^n \in \mathcal{M}$ must be computed. Here τ is a positive number which is related to the time step size.

Another example is to eddy current problems arising in the theory of magnetic field (c.f. [20]), which makes use of both Theorem 10.5 and 10.6. The details and other possible applications will be reported elsewhere.

10.3 Application to Finite Element Discretizations

In this section, we shall illustrate the application of the abstract theory and algorithms discussed in the previous section to a second order elliptic boundary value problem approximated using finite element functions on a quasi-uniform mesh. We first show that the hypotheses of the previous section are satisfied. We also consider the computational complexity of the resulting algorithm in both serial and parallel computing applications.

We shall consider the problem of approximating the solution u of

$$\begin{aligned} \mathcal{L}U &= f \quad in\, \Omega, \\ U &= 0 \quad on\, \partial\Omega. \end{aligned}$$

Here Ω is a bounded domain in $\mathbb{R}^d$ and $\mathcal{L}$ is given by (3.2) with the corresponding bilinear form $A(\cdot,\cdot)$ given by (3.3).

We assume that Ω can be partitioned with a nested sequence of quasi-uniform triangulations as is done in Section 4.1 that corresponds to a nested sequence of finite element spaces as follows:

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_j.$$

We seek the Galerkin approximation $u \in \mathcal{M}$ (to the solution U above defined by

$$A(u,v) = (f,v) \quad \forall v \in \mathcal{M}. \tag{10.20}$$

We take the inner products $(\cdot,\cdot) \stackrel{\text{def}}{=} (\cdot,\cdot)_{L^2(\Omega)}$ and $A(\cdot,\cdot)$ as defined above. Therefore we can use the abstract framework in the above section. the operators P_k, Q_k and A_k are the corresponding operators. We notice that in this case P_k is the standard Galerkin projection, Q_k is the orthogonal L^2 projection and A_k is the discretized $\mathcal{L}$ on $\mathcal{M}_k$.

The assumptions (**A10.1**) and (**A10.2**) are verified in this context by Proposition 3.5 and 3.6. The assumption (**A10.3**) is strongly tied to the elliptic regularity assumption (3.5) and in fact (**A10.3**) holds for exactly the same α as given in (3.5). The verification of (**A10.3**) is given by Proposition 4.1.

In order to discuss our preconditioner for the finite element equation discretizations, we need some other notation.

For each k, $\mathcal{M}_k$ has a natural basis, namely, the so-called nodal basis $\{\phi_i^k\}_{i=1}^{n_k}$ that satisfies

$$\phi_i^k(x_l) = \delta_{il}, \quad \forall i, l = 1, \ldots, n_k,$$

where $\{x_l : l = 1, \ldots, n_k\} = \mathcal{N}_k$ is the set of interior nodal points of the triangulation on which $\mathcal{M}_k$ is defined.

It is often convenient to use the scaled nodal basis as follows:

$$\{\bar{\phi}_i^k\} = \{h_k^{\frac{2-d}{2}} \phi_i^k\}.$$

By using these nodal bases, we can then modify the preconditioner given in (10.3) as follows:

$$\hat{B}v = A_1^{-1}Q_1 v + \sum_{k=2}^{j} \sum_i (v, \bar{\phi}_i^k)\bar{\phi}_i^k. \tag{10.21}$$

Lemma 10.5

$$(Bv, v) \asymp (\hat{B}v, v) \quad \forall v \in \mathcal{M}_j.$$

Proof. Note that, for $v \in \mathcal{M}_j$

$$(\hat{B}v, v) = (A_1^{-1}Q_1 v, v) + \sum_{k=2}^{j} \sum_i (Q_k v, \bar{\phi}_i^k)^2$$

and

$$(Bv, v) = (A_1^{-1}Q_1 v, v) + \sum_{k=2}^{j} \lambda_k^{-1} \|Q_k v\|^2.$$

Hence it remains to verify that

$$\|v_k\|^2 \asymp \lambda_k \sum_i (v_k, \bar{\phi}_i^k)^2 \quad \forall v_k \in \mathcal{M}_k.$$

To see this, writing $v_k = \sum \mu_i \phi_i^k = h_k^{\frac{d-2}{2}} \sum \mu_i \bar{\phi}_i^k$, the above inequality then becomes:

$$\sum_{i,j} (\bar{\phi}_i^k, \bar{\phi}_i^k) \mu_i \mu_j \asymp \lambda_k \sum_{i,j} \sum_l (\bar{\phi}_i^k, \bar{\phi}_l^k)(\bar{\phi}_j^k, \bar{\phi}_l^k) \mu_i \mu_j$$

which, in terms of the mass matrix $M_k = ((\bar{\phi}_i^k, \bar{\phi}_i^k))$ and the nodal value vector $\mu = (\mu_1, \dots, \mu_{n_k})^T$, can be rewritten as

$$< M_k \mu, \mu > \asymp \lambda_k < M_k^2 \mu, \mu > .$$

By Proposition 3.5

$$< M_k \mu, \mu > \asymp h_k^2 |\mu|^2.$$

and hence

$$< M_k^2 \mu, \mu > \asymp h_k^4 |\mu|^2.$$

The desired result then follows since $\lambda_k \asymp h_k^{-2}$ by Theorem 3.8. ■

Combining Theorem 10.2 with Lemma 10.5, we obtain

Theorem 10.7

$$\kappa(BA) \lesssim j^{\min(\frac{1}{\alpha}, 2)}$$

where $\alpha \in (0, 1]$ *is as in* (3.5).

We may avoid the inversion of A_1 by a further modified $\mathcal{B} : \mathcal{M} \mapsto \mathcal{M}$ defined, for $v \in \mathcal{M}$, by

$$\mathcal{B} v = \sum_{k=1}^{j} \sum_i (v, \bar{\phi}_i^k) \bar{\phi}_i^k. \tag{10.22}$$

For this modification, we have

Theorem 10.8

$$\kappa(\mathcal{B}A) \leq \begin{cases} (h_1^{-2} + j)j; \\ \\ (h_1^{\frac{-2\alpha}{1-\alpha}} + j)^{\frac{1-\alpha}{\alpha}} j. \end{cases}$$

Remark 10.5 We notice that, since $j = O(|\log h_j|)$, if $\alpha = 1$, $\kappa(BA) = O(|\log h_j|)$ and in any case not worse than $O(|\log h_j|^2)$ which is the condition number of the hierarchical basis preconditioner for $d = 2$. An important feature of this result is that it holds in any number of dimensions.

In the remainder of this section, we consider computational issues involved in implementing the above algorithm in serial and parallel computing architectures. However, before proceeding, we make the following observation. Even though we have defined $\mathcal{B}$ as an operator on $\mathcal{M}$, in a preconditioned iterative scheme we are only required to compute $\mathcal{B}v$ given the data $W_j^l = (v, \phi_j^l)$. This is because $v = A_j\theta$ and to avoid the solution of gram matrix problems, we always compute $\{(A_j\theta, \phi_j^l) = A(\theta, \phi_j^l)\}$ instead of $A_j\theta$ for a given $\theta \in \mathcal{M}$.

We first consider parallel implementation of the preconditioner $\mathcal{B}$. The execution of (1.4) can obviously be made parallel in many ways by breaking up the terms into various numbers of parallel tasks. The optimal splitting of the sum is clearly dependent on characteristics of the individual parallel computer, and for example, memory management considerations, task initialization overhead, the number of parallel processors, etc. We note, however, the simplicity of the form of (1.4) allows for almost complete freedom for parallel splitting.

It is of theoretical interest to consider the algorithm on a shared memory machine with an unlimited number of processors. As above, the implementation $\mathcal{B}v$ involves two steps, the calculation of the coefficients W_k^l and the computation of the representation of $\mathcal{B}v$ in the basis for $\mathcal{M}$. Each coefficient can be computed independently and involves a linear combination (not necessarily local) of the values of W_j. With enough processors, a linear combination of m numbers can be computed in $\log_2(m)$ time. Hence the coefficient vectors $\{W_k\}$ can be computed in $\log_2(N)$ time where N is the dimension of $\mathcal{M}$. Each coefficient of $\mathcal{B}v$ involves a linear combination of $M_n j$ contributions from the j grid levels (here M_n is the maximum number of neighbors for any given level. Thus, computation of $\mathcal{B}v$ can in done in time bounded by Cj.

We next consider the serial version of the algorithm. Let $v \in \mathcal{M}$ be given and define $W_k^l = (v, \phi_k^l)$. Let W_k denote the vector with entries $(W_k)_l = W_k^l$. We need to compute the action of $\mathcal{B}v$ given W_j. We define W_{k-1} from W_k in a recursive manner. Note that each basis function in $\mathcal{M}_{k-1}$ can be written as a local linear combination of basis functions for $\mathcal{M}_k$. Thus, each value of W_{k-1}^l can be written as a local linear combination of values of W_k. Moreover, the work involved in computing W_{k-1} from W_k is proportional to the number of unknowns in $\mathcal{M}_{k-1}$. Consequently, the work involved in computing the vectors $\{W_k\}$, $k = 1, \ldots, j$ bounded by a constant times the number of unknowns in $\mathcal{M}$. Once the vectors $\{W_k\}$ are known, we are left to compute the representation of $\mathcal{B}v$ in the basis for $\mathcal{M}$. To do this, we compute the representation of

$$\mathcal{B}_m v \equiv \sum_{k=1}^{m} \sum_{l} (v, \phi_k^l) \phi_k^l,$$

in the basis for $\mathcal{M}_m$, for $m = 1, \ldots, j$. The result at $m = j$ is of course the basis representation for $\mathcal{B}v$. For $m = 1$, the representation is already given by W_1. The representation of $\mathcal{B}_m v$, for $m > 1$ is calculated from that of $\mathcal{B}_{m-1}v$ by interpolating the $\mathcal{B}_{m-1}v$ results (i.e. expanding them in terms of the m'th basis) and adding the m'th level contribution from W_m. The work of calculating the representation of $\mathcal{B}_m v$ given that for $\mathcal{B}_{m-1}v$ is on the order of the number of unknowns in $\mathcal{M}_m$ and thus the total work for this algorithm is bounded by a constant times the number of unknowns on the finest grid.

Remark 10.6 The serial implementation of the operator $\mathcal{B}$ is closely related to the multigrid V-cycle algorithm. The step of computing W_{k-1} from W_k in $\mathcal{B}$ is nothing more that the step which "transfers the residuals" from grid level k to $k-1$ in a multigrid V-cycle algorithm. However, the multigrid algorithm requires extra computation since it must smooth and then compute new residuals on the k'th level before transferring. The second step in the serial algorithm for $\mathcal{B}$ is also duplicated in the "coarser to finer interpolation" step in the multigrid V-cycle algorithm. The symmetric multigrid V-cycle requires extra computation since it requires additional smoothing on each grid level. Thus, the serial $\mathcal{B}$ algorithm, in terms of complexity, is similar to a multigrid V-cycle algorithm without smoothing.

10.4 Application to Interface Problems with Large Jumps

The interface problem has been studied in the context of the ordinary multigrid method in Section 8.5 and 8.6, where we have essentially proved that the two-level multigrid algorithm converges uniformly with respect to the jumps in the coefficient. It is still unknown how the algorithm behaves in the general multilevel case. However for the hierarchical basis preconditioner, it is known that the condition number is not effected by the jumps if propers weights are used. In this section, we show that such a property also holds for our preconditioner in the two dimensional case and sometimes in three or higher dimensional cases as well.

To avoid the unnecessary repetition, we use the same model problem as given in Section 8.5 and the multilevel spaces are also as described there. In this way, the whole theory in the preceding section can be carried over here with the ordinary L^2 projection replaced by the weighted L^2 projection defined by (3.39). Correspondingly, (10.22) is replaced by the following

$$\mathcal{B}v = \sum_{k=1}^{j} \sum_{i} \omega_i^k (v, \bar{\phi}_i^k)_{L^2_\omega(\Omega)} \bar{\phi}_i^k. \tag{10.23}$$

where ω_i^k is the arithemetic average of those ω_l's such that $\bar{\Omega}_l$ contains the node x_i^k.

Theorem 10.9 *If all the cross points of the interfaces lie in $\partial\Omega$,*

$$\kappa(\mathcal{B}A) \lesssim |\log h_j|^2,$$

where $\hat{B}$ is defined by (10.23)*.*

Proof. In fact ($\mathbf{A}10.1'$) is given by Theorem 3.7. The desired result then follows. ■

We can remove the constraint on the location of the cross points in above theorem in two dimensional problems. Namely we have

Theorem 10.10 *If $d = 2$, then*

$$\kappa(\mathcal{B}A) \lesssim |\log h_j|^3,$$

where $\mathcal{B}$ is defined by (10.23)*.*

Proof. According to Theorem 4.3, ($\mathbf{A}10.1'$) holds with $K_1' = O(|\log h_j|)$. The desired result then follows from the general theory in Section 10.1. ■

Remark 10.6 In regard to Remark 3.1, the results in this section can be generalized and improved.

Bibliography

[1] Adams, R. *Sobolev Spaces*, Academic press, New York(1975).

[2] Asktrakhantsev, G. P., *An iterative method of solving elliptic net problems,* Z. Vvcis1. Mat. i. Mat. Fiz., 11(1971), 439–448.

[3] Aziz, A.k. and Babuška, I. *Part I, survey lectures on the mathematical foundations of the finite element method* in "The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations", (Ed. A.K. Aziz), Academic Press, New york, 1972, pp. 1–362.

[4] Axelsson, O., Barker, V.A. *Finite Element Solution of Boundary Value Problems: Theory and Computation;* Academic Press, New York 1984.

[5] Babuška, I., Kellogg, R. and Pitkäranta, J, *Direct and inverse error estimates for finite elments with mesh refinements,* Numer. Math., 33(1979), 447–471.

[6] Bakhvalov, N., *On the convergence of a relaxation method with natural constraints on the elliptic operator,* Zh. Vychisl. Mat. mat. Fiz., 6(1966), 861–885.

[7] Bank, R.E. *A comparison of two multilevel iterative methods for nonsymmetric and indefinite elliptic finite element equations*, SIAM J. Numer. Anal 18 (1981), 724–743.

[8] Bank, R.E. and Douglas, C.C. *Sharp estimates for multigrid rates of convergence with general smoothing and acceleration*, SIAM J. Numer. Anal. 22(1985), 617–633

[9] Bank, R.E. and Dupont, T.F., *Analysis of a two–level scheme for solving finite element equations,* Report CNA–159, Center for NumeriaAnalysis, University of Texas at Austin 1980.

[10] Bank, R.E. and Dupont, T.F., *An optimal order process for solving elliptic finite element equations,* Math. Comp. 36(1981), 967–975.

[11] Bank, R., Dupont, T. and Yserentant, H., *The hierarchical Basis Multigrid Method*, (preprint).

[12] Bank, R.E. and Scott, R.L., *On the conditioning of finite element equations with highly refined meshes,* (preprint)

[13] Bank, R.E., Sherman, A. and Weiser, A. *Refinement algorithms and data structures for local mesh refinement*, in Scientific Computing, R.Stepleman et al. eds., Amsterdam: IMACS/ NORTH HOLLAND 1983.

[14] Braess, D., *The contraction number of a multigrid method for solving the Poisson equation*, Numer. Math. 37(1981) 387–404.

[15] Braess, D., *The convergence rate of a mutigrid method woth Gauss–Seidel relaxation for the Poisson equation,* in [49].

[16] Braess, D., *The convergence rate of a mutigrid method woth Gauss–Seidel relaxation for the Poisson equation,* (revised), Math. Comp., 42(1984), 967–975.

[17] Braess, D., *On the combination of the multigrid method and conjugate gradient,* in [50], 53–64.

[18] Braess, D. and Hackbusch, W., *A new convergence proof for the multigrid method including the V–cycle,* SIAM J. Numer. Anal. 20(1983), 967–975

[19] Braess, D. and Verfürth, R., *Multigrid methods for non–conforming finite element methods* (preprint)

[20] Bramble, J.H. and Pasciak, J.E. *An efficient numerical procedure for the computation of state harmonic currents in flat plates,* preprint.

[21] Bramble, J.H. and Pasciak, J.E. *New convergence estimates for multigrid algorithms,* Math. Comp. 49 (1987), 311–329.

[22] Bramble, J.H., Pasciak, J.E. and Schatz, A.H., *An iterative method for elliptic problems by substructures*, Math. Comp. 46(1986), 361–369.

[23] Bramble, J.H.,Pasciak, J.E. and Schatz, A.H., *The Construction of Preconditioners for elliptic Problems by Substructuring. I*, Math. Comp. 47(1986),103–134.

[24] Bramble, J., Pasciak, J and Xu, J., *The analysis of multigrid algorithms for the nonsymmetric and indefinite problems*, Math. Comp., 51(1988), 389–414.

[25] Bramble, J., Pasciak, J and Xu, J., *The analysis of multigrid algorithms with nonnested and noninherited forms*, to appear in Math. Comp.

[26] Bramble, J., Pasciak, J and Xu, J., *Parallel multilevel preconditioners*, to appear in Math. Comp.

[27] Bramble, J., Pasciak, J and Xu, J., *A multilevel preconditioner for domain decomposition boundary systems*, (in preparation).

[28] Bramble, and Xu, J., *A new multigrid preconditioner*, MSI Workshop on Practical Iterative Methods for Large Scale Computations, Oct. 1988, Minneapolis.

[29] Brandt, A., *Multi–level adaptive technique for fast numerical solutions to boundary value problems*, Pro. 3rd Int. Conf. Numerical Methods in Fluid Mechanics (Paris, 1972); Lecture Notes in Physics 18, Springer–Verlag, Berlin, 82–89.

[30] Brandt, A., *Multi–level adaptive solutions to boundary–valued problems,* Math. Comp. 31(1977), 333–309

[31] Brandt, A., *Algebraic multigrid theory: the symmetric case*, Appl. Math. Comp. 19(1986), 23–56.

[32] Brenner, S., *An optimal order multigrid method for P_1 nonconforming finite elements*, Math. Comp., 52(1989), 1–15.

[33] Brenner, S., *An optimal order multigrid method for the biharmonic equation*, to appear in SIAM J. Numer. Anal.

[34] Brezzi, F. and Marini, L., *On the numerical solutions of plate bending problems by hybrid methods,* RFAIROSR Anal. Numer, 3(1975), 5–50.

[35] Chen, Quanmiao, *Private communication*

[36] Ciarlet, P.G., “The Finite Element Method for Elliptic Problems”, North–holland, New York, 1978.

[37] Crouzeix, M. and Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations I*, R.A.I.R.O., R-3(1973) 33-75.

[38] Crouzeix, M. and Thomee, T. *The stability in L_p and W_p^1 of the L_2–projection onto finite element function spaces* Math. Comp. 48(1987), 521–532.

[39] Decker, N., Mandel, J. and Parter, S., *On the role of regularity in the multigrid methods,* in [68], 143–156.

[40] Douglas, C. *Multigrid algorithms with applications to elliptic boundary–value problems* SIAM J.Numer. Anal. v.21, 2(1984), 236–254

[41] Dupont, T., Kendall, R.P. and Rachford, H.H., *An approximate factorization procedure for solving self–adjoint elliptic difference equations*, SIAM J.Numer. Anal 5 (1986), 559–573.

[42] Fedorenko, R.P., *A relaxing method for solving ellptic difference equations,* USSR Comput. Math. and Math. Phys. 1,5(1961), 1092–1096.

[43] Fedorenko, R.P., *The speed of convergence of one iterative process,* USSR Comput. Math. and Math. Phys. 4,3(1964), 227–235.

[44] Fried, I., *Condition of finite element matrices generated from nonuniform meshes,* AIAA J., 10(1971), 219–221.

[45] Grisvard, P., *Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain,* in "Numerical Solution of Partial Differential Equations, III" , Hubbard, B. ed, Academic Press, New York, 1976, 207–274.

[46] Hackbusch, W., *On the convergence of a multi-grid iterations applied to finite element equations,* Report 77-8, Universitat zu Koln, July, 1977.

[47] Hackbusch, W., *Mutigrid convergence theory*, in [49].

[48] Hackbusch, W., "Multi–Grid Methods and Applications", Berlin, Heidelberg, Springer–verlag, New york, 1985

[49] Hackbusch, W. and Trottenberg, U.(eds.), "Multigrid Methods" (Proceedings, Köln–Porz 1981), Lecture Notes in Math. 960, Springer–Verlag 1982.

[50] Hackbusch, W. and Trottenberg, U.(eds.), "Multigrid Methods II" (Proceedings, Cologne 1985), Lecture Notes in Math. 1228, Springer–Verlag 1986.

[51] Hilbert, S., *A molifier useful for approximations in Sobolev spaces and some applications to approximations of differential equations*, Math. Comp., 27(1973), 81–89.

[52] Kato, T., "Perturbation Theory for Linear Operators," Sprin–Verlag, New York, 1976.

[53] Kettler, R. *Analysis and comparison of relaxation schemes in robust multigrid and preconditioned conjugate gradient methods,*

[54] King, T., *On the construction of preconditioners by subspace decomposition*, (preprint).

[55] Kondrat'ev,V.A., *Boundary problems for elliptic equations with conical or angular Points,* Trans. Moscow Math. Soc. 16 (1967), 227–313.

[56] Kočvara, M. and Mandel, J., *A multigrid method for three–dimensional elasticity and algebraic convergence estimates,* Appl. Math. Comp. 23(1987), 121–135.

[57] Krein, S.G.and Petunin, Y.I. *Scales of Banach spaces,* Russian Math.Surveys 21 (1966), 85–160.

[58] Lions, J.L. and Magenes, E., "Problémes aux Limites non Homogénes et Applications", dunod, Paris, 1968

[59] Maitre, J.F. and Musy, F., *The Contraction Number of a Class of Two–Level Methods, an Exact Evaluation for some Finite Element Subspaces and Model Problems,* In [49].

[60] Maitre, J.F. and Musy, F., *Méthodes multigrilles: op'erateur associé et estimation de convergence; le case du V–cycle,* C. R. Acad. Sci. Paris. Sér. I, 296(1983), 521-524.

[61] Maitre, J.F. and Musy, F., *Multigrid methods: convergence theory in a variational framework,* SIAM J. Numer. Anal. 21(1984), 657-671.

[62] Maitre, J.F. and Musy, F., *Algebraic formalization of the multigrid method in the symmetric and positive definite case – a convergence estimation for the V–cycle,* in "Multigrid Mthods for Integral and Differential Equations", D.J. Paddon and H. Holstien(eds), Claridon Press, Oxford, 1985.

[63] Mandel, J., *Étude algébrique d'une méthode multigrille pour quelques problémes de frontiére libre,* C. R. Acad. Sci. Paris Sér. I. Math. 298(1984), 469–472.

[64] Mandel, J., *Multigrid convergence for nonsymmetric, infinite variational problems and one smoothing step,* in "Proc. copper Mtn. Conf. Multigrid Methods, " Applied Math. Comput., 1986, 201–216.

[65] Mandel, J., *Algebraic study of multigrid methods for symmetric, definite problems,* (preprint).

[66] Mandel, J., McCormick, S.F. and Bank, R., *Variational multigrid theory,* in "Multigrid Methods", Ed. S. McCormick, SIAM, Philadelphia, Penn. 1988. pp.131–178.

[67] Mandel, J., McCormick,S.F., and Ruge, J., *An algebraic theory for multigrid methods for variation problems,* SIAM J. Numer. Anal., 25(1988), 91-110.

[68] McCormick, S.(ed.), "Multigrid Methods: theory, application, and supercomputing," Lecture Notes in Pure and Applied Math. 110, Marcel Dekker, Inc. 1988.

[69] McCormick, S.F., *Multigrid methods for variational problems: further results,* SIAM J. Numer. Anal. 21(1984), 255–263.

[70] McCormick, S.F., *Multigrid methods for variational problems: general theory for the V–Cycle,* SIAM. Anal.22(1985), 634–643.

[71] McCormick, S. and Ruge, J.,, *Multigrid method for variational problems,* SIAM J. numer. Anal. 19(1982), 924-929.

[72] Musy, F., *Sur les méthodes multigrilles: formalisation algébrique et démonstration de convergence,* C. R. Acad. Sci. Paris. Sér. I, 295(1982), 471-474.

[73] Nečas, J., "Les Méthodes Directes en Théorie des Équations Elliptiques", Academia, Prague, 1967.

[74] Nicolaides, R.A., *On the* l^2 *convergence of an algorithm for solving finite element equations,* Math. Comp. 31(1977), 892–906.

[75] Nicolaides, R.A., *On multigrid convergence in indefinite case,* Math. Comp. 32(1978), 1082–1086.

[76] Rivara, M.C., *Algorithms for refining triangular grids suitable for adaptive and multigrid techniques,* Int. J. Numer. Methods Eng. 20(1984), 745–756.

[77] van Rosendale, J., *Rapid solution of finite element equations on locally refned grids by multilevel methods*, Ph.D. thesis, University of Illinois, 1980.

[78] Rudin, *Functional Analysis*

[79] Ruge, J. and Stüben ?? *Algebraic multigrid,* in "Multigrid Methods", Ed. S. McCormick, SIAM, Philadelphia, Penn. 1988. pp.131–178.

[80] Schatz, A.H., *An observation concerning Ritz–Galerkin methods with indefinite bilinear forms,* Math. Comp.28 (1974), 959–962.

[81] Scott, R. and Zhang, S., *Finite–element interpolation of nonsmooth functions satisfying boundary conditions,* (preprint, 1989)

[82] Smirnov, V., *A Course of High Mathematics*(English translation) v.5. Adison Wesley(1964).

[83] Verfürth, R., *The contraction number of a multigrid method with mesh ratio 2 for solving the Poisson equation*, Linear Algebra Appl., 60(1984), 332-348.

[84] Wahlbin, L., *On the sharpness of certain local estimates for H^1 projections into finite element spaces: influence of a reentrant corner,* Math. Comp., 42(1984), 1–8.

[85] Xu, J., *Some inequalities in Sobolev Spaces and finite and infinite elements on polygonal domains.* M. A. Thesis, Peking University, China (1984).

[86] Xu, J., *Error estimates and improved algorithms for infinite elements*, Proceeding of DD5 international conference, Beijing, China(1984).

[87] Xu, J. *Error estimates of the finite element method for the 2nd order elliptic equation with discontinuous coefficients*, J. Xianghtan Univ. No.1(1982)

[88] Xu, J. and Ying, L. *The analysis and improvement on infinite element algorithms*, Numer. Math. Sinica(1986) (in Chinese).

[89] Young, D., "Iterative solutions of large linear systems", Acad Press, New York 1971.

[90] Yserentant, H., *Hierarchical bases of finite element spaces in the discretization of nonsymmetric elliptic boundary value problems,* Computing 35(1985), 39–49.

[91] Yserentant, H., *Hierarchical bases give conjugate type methods a multigrid speed of convergence,* Appl. Math. and Comp. 19(1986), 147–358.

[92] Yserentant, H., *The convergence of multi–level methods for solving finite–element equations in the presence of singularities,* Math. Comp. 47(1986), 399–409.

[93] Yserentant, H., *On the convergence of multi–level methods for strongly nonuniform families of grids and any number of smoothing steps per level,* Computing, 30(1983), 305–313.

[94] Yserentant, H., *On the multi–level splitting of finite element spaces,* Numer. Math. 49(1986), 379–412.

[95] Yserentant, H., *On the multi–level splitting of finite element spaces for indefinite elliptic boundary value problems,* SIAM J.Numer. Anal. 23(1986), 581–595.

[96] Zhang, S., "Multi–level Iterative Techniques", Ph.D. thesis, Pennsylvania State University, 1988.

[97] Zhu, Q., *Pointwise and inner maximum norm estimates in the finite element method*, Numer. Math. Sinica, v.3, No.1(1981) pp.87–90.